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Instanton Sums and Monodromy

String theory

10-dim'l spacetimes

$X^6 \times M^4$ ← Minkowski

want supersymmetry (SUSY) in 4D.

$\Rightarrow$ covariantly constant spinor on X^6.

holonomy is restricted: $SU(3)$

$\rightsquigarrow$ Ricci flat Kähler,

$\Rightarrow$ X^6 is a projective alg. variety.

 $\Sigma \longrightarrow X^6$

worldsheet

- $\mathrm{Maps}(\Sigma, X^6)$.

quantum field theory

Expansion in q, coefficients are "instanton numbers".

want all maps in a fixed coho. class
with minimum action. $n_{\vec{d}}$

$$n_{\vec{0}} + \sum n_{\vec{d}}\, q^{\vec{d}} = \text{physical quantity}$$

e.g. $n_{\vec{d}}$ = Gromov-Witten invariants of X.

$X^6 \subseteq \mathbb{CP}^4$

$X^6 = \{ x_1^5 + x_2^5 + \cdots + x_5^5 = 0 \}$

QFT assoc. to X^6
can sometimes be described by a
"gauged linear sigma model".

Instantons are easy to calculate.

Related to original QFT by "renormalization".

renormalization ⇒ need a parameter change

(in q) when comparing answers.

Superspace

P. Deligne, D. Freed
"Supersolutions" in: Quantum Fields + Strings, vol 1

ed by Deligne et al, AMS 1998?

$\mathbb{Z}_2$-graded

A superspace is a supermanifold whose "body" is Minkowski space.

$$M^{2|(2,2)}$$

x^0, x^1 : coords on M^c

$(x^0)^2 - (x^1)^2$

$\partial_\pm = \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}$, even vector fields

odd vector fields: $D_+, \bar{D}_+, \bar{D}_-, D_-$

$$[D_+, D_+] = -\,- = 0$$

$$[D_-, D_-] = -\,- = 0$$

$$[D_+, \overline{D}_+] = -2\partial_+$$

$$[D_-, \overline{D}_-] = -2\partial_-$$

$\mathcal{L}$ = Super Lie algebra = extension of Poincaré alg.
by these odd v.f.

$M^{2|(2,2)}$ is acted upon by $\exp(\mathcal{L})$.

$D_\pm$, $\partial_\pm$ are left-invariant v.f. on $M^{2|(2,2)}$

$\Phi : M^{2|(2,2)} \longrightarrow \mathbb{C}^n$. Super field.

e.g. $D_+ \Phi \big|_{M^2}$, $D_- \Phi \big|_{M^2}$ this produces "usual" fields

$$\Phi \big|_{M^2} = \varphi : M^2 \to \mathbb{C}^n.$$

chiral superfield: $\bar{D}_+ \Phi_i = \bar{D}_- \Phi_i = 0.$

("holomorphic")

Need compact abelian group G.

$$\rho : G \to U(1)^n \subseteq GL(n, \mathbb{C}).$$

we will also need

$$\tau \in \mathrm{Hom}(\mathfrak{g}, \mathbb{C})$$

$W_{\cdot} =$ G-invariant poly on $\mathbb{C}^n$.

$$\mathcal{L} = \int d^4\theta \left(\| e^{-i\, d\rho \circ V} \|^2 - \frac{1}{4e^2} \|\Sigma_{\cdot} \right.$$

G-invariant
(kinetic terms)

$$d^4\theta = d\theta^+ \, d\theta^- \, d\bar{\theta}^+ \, d\bar{\theta}^-$$

V = gauge multiplet: $M^{2|(2,2)} \to \mathfrak{g}$

$\Sigma = \bar{D}_+ D_- V$.

"twisted chiral" $\quad \bar{D}_+ \Sigma = D_- \Sigma = 0$.

Need norm on $\mathfrak{g}$.

$$\mathcal{L} = \text{kinetic term}$$

$$+ \ \mathrm{Re} \int d\theta^+ d\theta^- \, W(\Phi_1, \ldots, \Phi_n)$$

$$+ \ \mathrm{Im} \int d\theta^+ d\bar{\theta}^- \, (\tau, \Sigma)$$

$$\tau \in \mathrm{Hom}(\mathfrak{g}, \mathbb{C})$$

W = poly. Coefs of W. = params

$\rho : G \to U(1)^n$.

$r \in \mathrm{Hom}(\mathfrak{g}, \mathbb{C})$ more params.

Example $W(x_0, x_1, \ldots, x_5)$

$$= x_0 x_1^5 + \cdots + x_4 x_5^5 + x_0 x_1 x_2 x_3 x_4 x_5$$

$G = U(1)$

x_0	x_1	$\cdots$	x_5
-5	1	$\cdots$	1

$\leftarrow Q_i^a$

$(t, \mapsto t^{-5} x_0 \quad t x_1 \quad \cdots)$

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Low-energy analysis

Pick a basis for $\mathfrak{g}$

$V = (V_1, \dots, V_{n-d})$.

group action $\quad e^{V_a} : \Phi_i \to (e^{V_a})^{Q_i^a} \Phi_i$

Derivatives of certain component fields do not occur in action; can solve:

$$\tau = ir + \frac{1}{2\pi}\theta.$$

comp of r_a

$$D_a = -e^2\left(\sum Q_i^a|\varphi_i|^2 - r_a\right)$$

$$F_i = -\frac{\partial W}{\partial \varphi_i}$$

comp of Φ_i

Potential for bosonic fields:

$$U = \frac{1}{2e^2}\sum D_a^2 + \sum |F_i|^2 + \cdots$$

To minimize U, must have

$$D_a = 0, \quad F_i = 0.$$

Example

$$-5|\varphi_0|^2 + |\varphi_1|^2 + \dots + |\varphi_5|^2 - r = 0.$$

$$W(\vec{x}) = x_0 P_5(\vec{x})$$

$$\frac{\partial W}{\partial x_0} = P_5(\vec{x}) \qquad = 0$$

$$\frac{\partial W}{\partial x_i} = x_0 \frac{\partial P_5}{\partial x_i}(\vec{x}) = 0$$

if $r < 0$ then not all $\phi_1, \ldots, \phi_5$ can vanish.

$$\mathbb{C}^\times \times \mathbb{C}^5 \setminus \{0\} \cong \mathbb{C}^\times \times \underset{\mathbb{C}\mathbb{P}^4}{\boxed{S^9/U(1)}}$$

get a complex line bundle $\mathcal{O}(-5)$ over $\mathbb{C}\mathbb{P}^4$.

If ψ is generic,

P, $\dfrac{\partial P}{\partial x_i}$ will share no common zero outside $\vec{0} \in \mathbb{C}^5$.

$$\Rightarrow \quad P = 0, \quad x_0 = 0.$$

$\mathbb{CP}^4$

$\Rightarrow \quad p = 0, \; x_0 = 0.$

$\mathbb{CP}^4$

(quintic hyp) $\subseteq \mathbb{CP}^4$

(Calabi-Yau).

Constraint on choices

$\rho : G \to U(1)^n$, $W(x_1, \ldots, x_n)$, G-invariant

family of theories with coefs varying

{monomials in W}.

$x^{t_\alpha} = \prod x_i^{t_{\alpha i}}$

t_a = row of a nonnegative integer matrix T.

G ~~takes~~ determines a set of G-invariant Laurent monomials.

P has rows P_{ai} integers.

$$x^{P_a} = \prod x_i^{P_{ai}}$$

$G \subseteq U(1)^m$.

$$T = S P.$$

$\uparrow$

nonsingular
int

rank d

Mirror theory:

$$t_T = t_P + t_S.$$

new moment.

new gluon

conformally invariant theory in IR.:

$\exists\ \vec{\mu}, \vec{\nu}$ rational vectors

$$T\vec{\mu} = (1\,1\,1\,1 \cdots)^T$$

$$\vec{\nu}^T T = (1\,1\,1\,1 \cdots 1)$$