Zeta Functions of Toric Calabi-Yau Hypersurfaces

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Contents

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1 Toric Geometry

1.1 n-Torus

Denote by $\mathbb{G}_m^n$ the algebraic n-torus over $\mathbb{F}_q$. Notice that its $\mathbb{F}_{q^k}$-rational points are $\mathbb{G}(\mathbb{F}_{q^k}) = (\mathbb{F}_{q^k}^*)^n$ and so $\#\mathbb{G}_m^n(\mathbb{F}_{q^k}) = (q^k - 1)^n$. It follows that its zeta function is rational:

$$\begin{aligned} Z(\mathbb{G}_m^n/\mathbb{F}_q, T) : &= \exp(\sum_{k=1}^{\infty} \frac{\#\mathbb{G}_m^n(\mathbb{F}_{q^k})}{k} T^k) \\ &= \prod_{i=0}^{n} (1 - q^i T)^{(-1)^{n-i-1}\binom{n}{i}}. \end{aligned}$$

1.2 Basic problem

Given a Laurant polynomial $f(x_1, \cdots, x_n) \in \mathbb{F}_q[x_1^{\pm}, \cdots, x_n^{\pm}]$, we may define an affine toric hypersurface

$$U_f := \{x \in \mathbb{G}_m^n | f(x) = 0\} \hookrightarrow \mathbb{G}_m^n.$$

Wanting to understand the sequence of integers obtained by counting the $\mathbb{F}_{q^k}$-rational points of U_f leads to its zeta function:

$$Z(U_f/\mathbb{F}_q, T) = \exp(\sum_{k=1}^{\infty} \#U_f(\mathbb{F}_{q^k}) \frac{T^k}{k}) \in 1 + T\mathbb{Z}[[T]].$$

As with the n-torus, we wonder whether this too will be a rational function. Indeed, Dwork has shown this to be true.

Theorem 1.1 (Dwork). $Z(U_f/\mathbb{F}_q, T) \in \mathbb{Q}(T)$.

A consequence of this theorem is the existence of a formula for the numbers $\#U_f(\mathbb{F}_{q^k})$ in terms of the zeros and poles of the zeta function. However, how well do we know these zeros and poles? Knowing their precise values seems to be too difficult in general, so, we may ask for weaker results concerning their absolute values (p-adic and over $\mathbb{C}$). Also, we may wonder how these zeros and poles vary in a family.

1.3 Projective toric hypersurfaces

Let K be a field. With our Laurant polynomial f, write $f(x) = \sum_{j=1}^{J} a_j x^{V_j}$ where $a_j \in K$ and $V_j = (v_{1j}, \cdots, v_{nj}) \in \mathbb{Z}^n$. Associated to f is its Newton polytope:

$$\Delta(f) := \Delta := \text{ the closed convex hull of the } \{V_j\}\text{'s in } \mathbb{R}^n.$$

We will assume $\dim \Delta = n$. The Newton polytope will be used to define a graded algebra S_Δ as follows. First, define the the polytope $\bar{\Delta} \subset \mathbb{R}^{n+1}$, which is one dimension higher than Δ, as the closed convex hull of the origin in $\mathbb{R}^{n+1}$ and the points $(1, V_j) \in \mathbb{R}^{n+1}$. Next, define the cone $C(\overline{\Delta})$ as the cone generated by $\overline{\Delta}$. Observe that $C(\bar{\Delta}) = \bigcup_{k=1}^{\infty} k\overline{\Delta}$. Next, define the monoid $L(\overline{\Delta})$ as the lattice points in the cone $C(\overline{\Delta})$. It may be shown that $L(\bar{\Delta})$ is a finitely generated monoid. Finally, define the K-algebra

$$S_\Delta := K[L(\overline{\Delta})].$$

This means S_Δ consists of all finite sums of $a_u x^u$ where $a_u \in K$ and $u \in L(\bar{\Delta})$. Since $L(\bar{\Delta})$ is a finitely generated monoid, S_Δ is a finitely generated K-algebra. Further, we may define a grading on S_Δ by $deg(x^u) := u_0$ where $u = (u_0, \ldots, u_n)$. Therefore,

$$S_\Delta = \bigoplus_{d=0}^{\infty} (S_\Delta)_d$$

where $(S_\Delta)_d$ is the K-submodule of S_Δ consisting of all elements of S_Δ of degree d.

Since S_Δ is a finitely generated graded K-algebra, we may define a K-scheme $\mathbb{P}_\Delta := \mathrm{Proj}\ \mathrm{S}_\Delta$. This is the toric variety associated to Δ. Observe that this toric variety only depends on those terms of f that lie on the vertices of Δ. So, we may think of $\mathbb{P}_\Delta$ as an analogue of projective space. That is, since $x_0 f \in (S_\Delta)_1$, we may define $\overline{U_f} := \mathrm{Proj}\ \mathrm{S}_\Delta/(\mathrm{x}_0\mathrm{f})$. Notice that $\bar{U}_f$ embeds in $\mathbb{P}_\Delta$ by construction. Thus, we call $\bar{U}_f$ a toric hypersurface in $\mathbb{P}_\Delta$. It follows that we have the diagram:

$$\begin{array}{ccc} \overline{U}_f & \longrightarrow & \mathbb{P}_\Delta \\ \text{projective closure}\uparrow & & \uparrow \text{compactification w.r.t. } \Delta \\ U_f & \longrightarrow & \mathbb{G}_m^n. \end{array}$$

This raises the new questions: what is $Z(\overline{U_f}, T)$ and how is it related to $Z(U_f, T)$?

1.4 Δ-regularity

In this section, we define the notion of a Δ-regular polynomial f.

Let $\tau \subset \Delta$ be a face of the polytope of any dimension ranging between zero and n. Define the restriction of f to τ as $f^\tau = \sum_{V_j \in \tau} a_j x^{V_j}$. Using the operator $E_i := x_i \frac{\partial}{\partial x_i}$, define $f_i := E_i f$ for each $i = 1, \cdots, n$.

Definition 1.2. f is called Δ-*regular* if for each face $\tau \in \Delta$ of any dimension, the system

$$f^\tau = f_1^\tau = \cdots = f_n^\tau = 0$$

has no common solutions in $\mathbb{G}_m^n(K^{\text{alg. clos.}})$.

We may reformulate the definition of Δ-regularity as follows. Define $F := x_0 f - 1 \in S_\Delta$. Notice that

$$F_i := E_i F = x_i \frac{\partial F}{\partial x_i} = \begin{cases} x_0 f, & i = 0 \\ x_0 f_i & i = 1, \cdots, n \end{cases}$$

and $F_i \in (S_\Delta)_1$. For each i, define $U_{F_i} = \mathrm{Proj} S_\Delta/(F_i)$.

Proposition 1.3. f is Δ-regular if and only if $\bigcap_{i=0}^n U_{F_i} = \emptyset$.

1.5 Homological formulation of Δ-regularity

Each $F_i \in S_\Delta$ acts on S_Δ by multiplication:

$$\begin{aligned} F_i : S_\Delta &\to S_\Delta \\ g &\mapsto F_i g \end{aligned}$$

$F_i F_j = F_j F_i$.

Let $K.(S_\Delta, F_0, \cdots, F_n)$ be the Koszul complex

$$0 \longrightarrow S_\Delta \, e_0 \wedge \cdots \wedge e_n \xrightarrow{\partial} \cdots \xrightarrow{\partial} \bigoplus_{i=0}^n S_\Delta \, e_i \xrightarrow{\partial} S_\Delta \longrightarrow 0$$

$$\partial(a \, e_{i_1} \wedge \cdots \wedge e_{i_j}) = \sum_{k=1}^{j} (-1)^k F_{i_k}(a) \, e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_j}$$

$\mathrm{H}_0(K.) = S_\Delta/(F_0, F_1, \cdots F_n) = R_f$, the Jacobian ring of f.

Proposition 1.4. TFAE (the following are equivalent):

1) f is Δ-regular.

2) $\{F_0, F_1, \cdots, F_n\}$ forms a regular sequence of S_Δ.

3) $\mathrm{H}_i(K.) = 0, \forall \, i \geq 1$.

4) $\dim_K \mathrm{H}_0(K.) < \infty$.

5) $\dim_K \mathrm{H}_0(K.) = d(\Delta) = n! \, \mathrm{Vol}(\Delta) \in \mathbb{Z}_{>0}$.

1.6 Hodge numbers

Definition 1.5. Let $\Delta \subseteq \mathbb{R}^n$ be n-dimensional integral convex in $\mathbb{R}^n$. Let

$$W_\Delta(k) = \#(\mathbb{Z}^n \cap k\Delta) = \dim_K (S_\Delta)_k.$$

and

$$\sum_{k=0}^{\infty} W_\Delta(k) T^k, \quad \text{the Poincare series of } S_\Delta.$$

Definition 1.6. Define

$$h_\Delta(k) = \dim_K (R_f)_k$$

and

$$\sum_{k \geq 0} h_\Delta(k) T^k, \quad \text{the Poincare series of } R_f,$$

where f is Δ-regular and

$$R_f = S_\Delta \big/ (F_0, F_1, \cdots, F_n), \dim R_f = \mathrm{d}(\Delta) = n! \operatorname{Vol}(\Delta).$$

$$\Rightarrow \quad (1-T)^{n+1} \sum_{k \geq 0} W_\Delta(k) T^k = \sum_{k \geq 0} h_\Delta(k) T^k, \quad \text{ of degree } \leq n.$$

$$h_\Delta(k) = W_\Delta(k) - \binom{n+1}{1} W_\Delta(k-1) + \binom{n+1}{2} W_\Delta(k-2) + \cdots$$

Theorem 1.7 (Ehrhart). There exists a polynomial $\Lambda(t)$ of degree n such that

1) for $k \in \mathbb{Z}_{\geq 0}$, $W_\Delta(k) = \Lambda(k)$;

2) for $k \in \mathbb{Z}_{>0}$, $W_\Delta(k)^* := \#\{\text{interior lattice points in } k\Delta\} = (-1)^n \Lambda(k)$

$$(\Rightarrow (1-T)^{n+1} \sum_{k=0}^{\infty} W_\Delta^*(k) T^k = \sum_{k \geq 0} h_\Delta^*(k) T^k, \quad \text{a polynomial of degree} \leq n{+}1);$$

3) duality: $h_\Delta^*(k) = h_\Delta(n+1-k), k = 0, 1, \cdots, n+1$.

Proposition 1.8.

$$f, \Delta\text{-regular over } \mathbb{C} \Rightarrow h^k(P\mathrm{H}_c^{n-1}(U_f)) = h_\Delta(k+1).$$

Definition 1.9. Let $HP(\Delta)$ denote the Hodge polygon in $\mathbb{R}^2$ with vertices $(0,0)$ and $(\sum_{k=0}^m h_\Delta(k), \sum_{k=0}^m k h_\Delta(k)), m = 0, 1, \cdots, n$.

1.7 Reflexive Δ and Calabi-Yau hypersurface

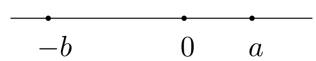
Definition 1.10. Let $\Delta \subseteq \mathbb{R}^n$, convex, integral, n-dimensional. Assume O is in the interior of Δ. Define

$$\Delta^* = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i y_i \geq -1, \forall\, (y_1, \cdots, y_n) \in \Delta \right\}.$$

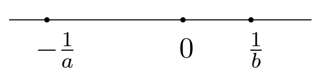
Δ^* is also an n-dimensional convex polytope, not necessarily integral. Clearly, $(\Delta^*)^* = \Delta$.

Definition 1.11. Δ is called reflexive if Δ^* is also integral.

<u>**Example**</u> $\Delta_{a,b}$:



$\Longrightarrow$ $\Delta^*_{a,b}$:



$\Delta_{a,b}$ is reflexive iff $a, b = 1$.

Definition 1.12. Let W be an irreducible normal n-dimensional projective variety with Gorenstein canonical singularities. Then W is called a *Calabi-Yau variety* if

1) the dualizing sheaf $\hat{\Omega}^n_W = O_W$ is trivial;

2) $\mathrm{H}^i(W, O_W) = 0, \forall\, 0 < i < n$.

Elliptic curves and $K3$-surfaces are CY.

Theorem 1.13 (Hibi, Batyrev). TFAE:

1) Δ is reflexive.

2) For any hyperplane $H = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n a_i x_i = 1\}$ such that $H \cap \Delta$ is a codimension 1 face of Δ, we have $a_i \in \mathbb{Z}$.

3) Hodge numbers are symmetric: $h_\Delta(k) = h_\Delta(n-k), 0 \leq k \leq n$.

4) The closure $\overline{U_f}$ of U_f (f is Δ-regular) in $\mathbb{P}_\Delta$ is a CY variety with canonical singularities.

Definition 1.14. For Δ reflexive; f Δ-regular, U_f is called an affine toric CY hypersurface.

Definition 1.15. Denote

$$M_p(\Delta) = \{f/\overline{\mathbb{F}}_p \mid \Delta(f) = \Delta,\ f \text{ is } \Delta\text{-regular}\}.$$

Let Δ be reflexive. The family $\{f \in M_p(\Delta)\}$ is called the mirror family of $\{g \in M_p(\Delta^*)\}$ over $\mathbb{F}_p$.

Question: If g is the "mirror" of f, $\quad Z(f/\mathbb{F}_q, T) \leftrightsquigarrow Z(g/\mathbb{F}_q, T)$?

Definition 1.16. A reflexive Δ in $\mathbb{R}^n$ is called *Fano*, if

1) Δ is simplicial, i.e., each codimension 1 face of Δ is a simplex. And
2) The vertices of each codimension 1 face of Δ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ in $\mathbb{R}^n$.

Proposition 1.17. Reflexive Δ is Fano $\Longleftrightarrow \mathbb{P}_{\Delta^*}$ is smooth.

1.8 A basic example

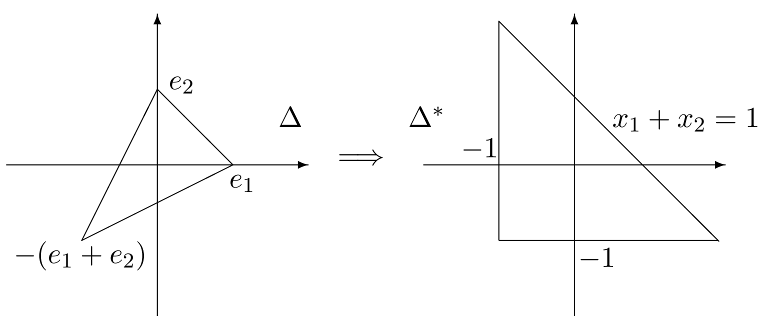
Take

$$\Delta = \langle e_1, e_2, \cdots, e_n, -(e_1 + \cdots + e_n) \rangle,$$

where e_i's are the standard unit vectors in $\mathbb{R}^n$. Then

$$\Delta^* = \langle (n, -1, \cdots, -1), (-1, n, \cdots, -1), \cdots, (-1, \cdots, -1, n), (-1, -1, \cdots, -1) \rangle.$$

Δ is reflexive, Δ Fano, but Δ^* NOT Fano if $n > 1$. For $n = 2$,



Let

$$f(\lambda, x) = x_1 + x_2 + \cdots + x_n + \frac{1}{x_1 x_2 \cdots x_n} - \lambda.$$

It's clear that $\Delta(f) = \Delta$.

$f(\lambda, x)$ is Δ-regular $\Longleftrightarrow \lambda \neq (n+1)\alpha, \alpha^{n+1} = 1$.

Mirror family:

$$\begin{aligned} g(\lambda, x) =& \frac{x_1^{n+1}}{x_1 x_2 \cdots x_n} + \cdots + \frac{x_n^{n+1}}{x_1 x_2 \cdots x_n} + \frac{1}{x_1 x_2 \cdots x_n} - \lambda \\ =& \frac{1}{x_1 x_2 \cdots x_n} (x_1^{n+1} + \cdots + x_n^{n+1} + 1 - \lambda x_1 x_2 \cdots x_n) \\ \overset{x_i \neq 0}{\leftrightsquigarrow}& 1 + x_1^{n+1} + \cdots + x_n^{n+1} - \lambda x_1 x_2 \cdots x_n \end{aligned}$$

Projective closure in $\mathbb{P}^n$:

$$x_0^{n+1} + x_1^{n+1} + \cdots + x_n^{n+1} - \lambda x_0 x_1 x_2 \cdots x_n = 0$$

(the well known family of CY hypersurfaces in $\mathbb{P}^n$.)

Let

$$G = (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1} = \left\{ (\zeta^{(1)}, \cdots, \zeta^{(n)}) \,\middle|\, (\zeta^{(i)})^{n+1} = 1, \prod_{i=1}^{n} \zeta^{(i)} = 1 \right\}.$$

Then G acts on $U_{g(\lambda,x)}$:

$$(\zeta^{(1)}, \cdots, \zeta^{(n)})(x_1, \cdots, x_n) = (\zeta^{(1)} x_1, \cdots, \zeta^{(n)} x_n).$$

Proposition 1.18. *$U_{f(\lambda,x)} = U_{g(\lambda,x)}/G$.*

Proof. If $g(\lambda, x) = 0$ for some $x, x_i \neq 0$, let

$$\begin{cases} y_1 = & x_1^{n+1}/x_1 \cdots x_n \\ & \vdots \\ y_n = & x_n^{n+1}/x_1 \cdots x_n \end{cases} \Rightarrow \begin{cases} x_1 \cdots x_n = y_1 \cdots y_n \\ \quad x_i^{n+1} = y_i y_1 \cdots y_n \end{cases}$$

$$\Rightarrow y_1 + \cdots + y_n + \frac{1}{y_1 \cdots y_n} - \lambda = 0.$$

$\square$

<u>**Exercise**</u>:
$\Delta = \Delta(x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} - \lambda) \Rightarrow h_\Delta(0) = h_\Delta(1) = \cdots = h_\Delta(n) = 1$. (Betti number $d(\Delta) = n + 1$.)

2 Zeta Functions

2.1 L-functions of exponential sums

For $f \in \mathbb{F}_q[x_1^{\pm}, \cdots, x_n^{\pm}]$, $U_f = \{f = 0\} \hookrightarrow \mathbb{G}_m^n$, we have

$$Z(U_f, T) = \exp(\sum_{k=1}^{\infty} \# U_f(\mathbb{F}_{q^k}) \frac{T^k}{k}).$$

Let

$$\Psi : \mathbb{F}_p \to \mathbb{C}^*$$
$$x \mapsto \psi(x) = \exp(\frac{2\pi i x}{p})$$

be a nontrivial character of $\mathbb{F}_p$. Then

$$\Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p} : \mathbb{F}_{q^k} \to \mathbb{C}^*$$

induces a nontrivial character of $\mathbb{F}_{q^k}$.

The exponential sum

$$S_k(x_0 f) = \sum_{x_i \in \mathbb{F}_q^*} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}(x_0 f).$$

It's easy to compute

$$\begin{aligned} q^k \# U_f(\mathbb{F}_{q^k}) = & \sum_{\substack{x_i \in \mathbb{F}_{q^k}^* \\ 1 \le i \le n}} \sum_{x_0 \in \mathbb{F}_q} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}(x_0 f) \\ =& (q^k - 1)^n + S_k(x_0 f) \\ =& \# \mathbb{G}_m^n(\mathbb{F}_{q^k}) + S_k(x_0 f). \end{aligned}$$

Then

$$Z(U_f, qT) = Z(\mathbb{G}_m^n, T) L(x_0 f, T),$$

where

$$L(x_0 f, T) = \exp(\sum_{k=1}^{\infty} \# S_k(x_0 f) \frac{T^k}{k}).$$

$\Rightarrow$It is enough to study $L(x_0 f, T)$.

2.2 Dwork's p-adic analytic character

Consider the Artin-Hasse series

$$t + \frac{t^p}{p} + \frac{t^{p^2}}{p^2} + \cdots .$$

The Newton polygon of this tells us that there are exactly $p-1$ roots of this series of slope $\frac{1}{p-1}$. Let π be one of these roots, and so $ord_p(\pi) = \frac{1}{p-1}$. Using this, we may define a *splitting function*

$$\theta(t) := \exp\left((\pi t) + \frac{(\pi t)^p}{p} + \cdots \right) \in \mathbb{Q}_p(\pi)[[T]].$$

Since

$$\exp\left(t + \frac{t^p}{p} + \cdots \right) = \prod_{(k,p)=1} (1 - t^k)^{-\frac{\mu(k)}{k}},$$

it follows that $\theta(t)$ converges on $|t|_p < p^{\frac{1}{p-1}}$. In particular, θ is defined at the Teichmüller points in $\mathbb{C}_p$. Splitting functions have the following remarkable properties:

Property 1. $\theta(1)$ is a primitive p-th root of unity.

Property 2. We may define a nontrivial additive character

$$\psi_k : \mathbb{F}_{p^k} \to \mathbb{C}_p^* \quad \text{by} \quad \psi_k(\bar{x}) := \theta(x)\theta(x^p)\cdots\theta(x^{p^{k-1}}) = \psi_1(\mathrm{Tr}_{\mathbb{F}_{p^k}/\mathbb{F}_p}(\bar{x})).$$

where x is the Teichmüller representative of $\bar{x}$.

2.3 Analytic representation of $S_k(x_0 f)$

Write $x_0\bar{f}(x) = \sum_{j=1}^{J} \bar{a}_j x_0 x^{v_j} \in \mathbb{F}_q[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then, with $q = p^a$, we have

$$\begin{aligned}
S_k(x_0 f) &= \sum_{\bar{x}_i \in \mathbb{F}_{q^k}^*} \psi_k(x_0 \bar{f}(\bar{x})) \\
&= \sum_{\bar{x}_i \in \mathbb{F}_{q^k}^*} \prod_{j=1}^{J} \psi_k(\bar{a}_j x_0 x^{v_j}) \\
&= \sum_{x_i^{q^k-1}=1, x_i \in \bar{\mathbb{Q}}_p} \prod_{j=1}^{J} \prod_{i=0}^{ak-1} \theta((a_j x_0 x^{v_j})^{p^i}) \qquad (1) \\
&= \sum_{x_i^{q^k-1}=1} F_a(f,x) F_a(f,x^q) \cdots F_a(f, x^{q^{k-1}}) \qquad (2)
\end{aligned}$$

where we have lifted the coefficients of f to $\mathbb{C}_p$, that is, $a_j = Teich(\bar{a}_j)$, and,

$$F_a(f,x) := \prod_{i=0}^{a-1} \prod_{j=1}^{J} \theta(a_j x_0 x^{v_j})^{p^i}.$$

This is the p-adic analytic representation of $S_k(x_0 f)$ that we will use.

2.4 Frobenius endmorphism

Recall S_Δ from section 1.3, with K replaced by $\mathbb{Z}_p$. Now, define

$$S_{\Delta,p} := \{ \sum_{u \in L(\bar{\Delta})} A_u \pi^{u_0} x^u | A_u \in \mathbb{Z}_p, A_u \to 0 \}.$$

Note, $S_{\Delta,p}$ is isomorphic to the p-adic completion of S_Δ at p. Now, with a norm defined by $\| \sum A_u \pi^{u_0} x^u \| := \sup_u |A_u|_p$, we see that $S_{\Delta,p}$ is a Banach $\mathbb{Z}_p$-module. By construction, we see that $\Gamma := \{\pi^{u_0} x^u | u \in L(\bar{\Delta})\}$ is an orthonormal basis for $S_{\Delta,p}$, that is, the coefficients tend to zero.

Consider the field $\mathbb{Q}_q(\pi)$ and its Galois group over $\mathbb{Q}_p(\pi)$, which is cyclic of order a generated by τ. By definition, τ sends Teichmüller representatives to their p-th power.

Using notation from the last section, define

$$F(f,x) := \prod_{j=1}^{J} \theta(a_j x_0 x^{v_j})$$

and

$$G(x) := F(f,x)F^{\tau}(f,x^p)F^{\tau^2}(f,x^{p^2})\cdots.$$

On the space $S_{\Delta,p} \otimes \mathbb{Z}_q[\pi]$, define the compact operators

$$\phi_1 := \psi_p \circ F(f,x)$$

and

$$\phi_a := \psi_q \circ F_a(f,x)$$

where, $q = p^a$, and

$$\psi_p(\sum A_u x^u) := \sum A_{pu}^{\tau^{-1}} x^u.$$

Note, we may formally write

$$\phi_1 = G(x)^{-1} \circ \psi_p \circ G(x) \qquad \text{and} \qquad \phi_a = G(x)^{-1} \circ \psi_q \circ G(x),$$

where

$$\psi_q(\sum A_u x^u) := \sum A_{qu} x^u.$$

2.5 Rationality of $L(x_0 f, T)$ and $Z(U_f/\mathbb{F}_q, T)$

Now ϕ_a has the following amazing property called the *Dwork trace tormula*:

$$S_k(x_0 f) = (q^k - 1)^{n+1} Tr(\phi_a^k)$$

where Tr denotes the trace of the operator. Recall the relation

$$\frac{1}{det(I - \phi_a T)} = \exp \sum_{k \geq 1} \frac{Tr(\phi_a^k)}{k} T^k.$$

Combining these with the binomial theorem, we see that

$$\begin{aligned} L(x_0 f, T) &= \exp \sum_{k \geq 1} \frac{S_k(x_0 f)}{k} T^k \\ &= \prod_{i=0}^{n+1} \left[det(I - q^i \phi_a T) \right]^{(-1)^{n-i}\binom{n+1}{i}}. \end{aligned}$$

This looks like rationality, however, remember that the operator ϕ_a acts on $S_{\Delta,p} \otimes \mathbb{Z}_q[\pi]$, an infinite dimensional space and so the characteristic polynomials are actually power series. However, since this operator is compact, $\det(I - q^i \phi_a T)$ is a p-adic entire function. Therefore, the L-function is p-adic meromorphic.

To prove rationality, we need to use an extension of a theorem of Borel proven by Dwork.

Theorem 2.1 (Borel). Let $g(T) \in \mathbb{Z}[[T]]$. Then $g(T) \in \mathbb{Q}(T)$ if and only if $g(T)$ satisfies both

1. $g(T)$ converges in some neighborhood of the origin in $\mathbb{C}$.

2. $g(T)$ is p-adic meromorphic.

We obtain

Theorem 2.2 (Dwork). $L(x_0 f, T) \in \mathbb{Q}(T)$ and so $Z(U_f/\mathbb{F}_q, T) \in \mathbb{Q}(T)$.

To prove this, we need only show that $L(x_0 f)$ converges in some neighbourhood of the origin in $\mathbb{C}$. Now,

$$|S_k(x_0 f)|_{\mathbb{C}} \leq (q^k - 1)^{n+1} \leq q^{k(n+1)}$$

and since

$$\sum_{k \geq 1} \frac{q^{k(n+1)}}{k} T^k$$

converges for $|T|_{\mathbb{C}} < 1/q^{n+1}$, we see that $L(x_0 f, T)$ converges for any $|T|_{\mathbb{C}} < 1/q^{n+1}$. This proves the theorem.

2.6 p-adic Cohomological formula for $L(x_0 f, T)$

As mentioned in section 2.5, we may define a compact operator ϕ_a on a p-adic Banach module $B := S_{\Delta,p} \otimes \mathbb{Z}_q[\pi]$. We may also define differential operators

$$D_i := G(x)^{-1} \circ x_i \frac{\partial}{\partial x_i} \circ G(x)$$

for each $i = 0, 1, \ldots, n$ acting on B. Since these commute, we may create a Koszul complex $K_\bullet(B, D_0, \ldots, D_n)$, the top line of the commutative diagram below. Also, since $\phi_a \circ D_i = q D_i \circ \phi_a$, we may define a chain map between complexes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & B^{\binom{n+1}{n+1}} \arrow[r, "d"] \arrow[d, "q^{n+1}\phi_a"'] & B^{\binom{n+1}{n}} \arrow[r, "d"] \arrow[d, "q^n \phi_a"'] & \cdots \arrow[r, "d"] \arrow[d] & B^{\binom{n+1}{0}} \arrow[r] \arrow[d, "\phi_a"] & 0 \arrow[d] \\
0 \arrow[r] & B^{\binom{n+1}{n+1}} \arrow[r, "d"] & B^{\binom{n+1}{n}} \arrow[r, "d"] & \cdots \arrow[r, "d"] & B^{\binom{n+1}{0}} \arrow[r] & 0
\end{tikzcd}$$

where

$$B^{\binom{n+1}{i}} := B \otimes \Lambda^i(\oplus_{j=0}^n \mathbb{Z} e_j)$$

and $d : B^{\binom{n+1}{i}} \to B^{\binom{n+1}{i-1}}$ is defined by

$$d(a e_{j_1} \wedge \cdots \wedge e_{j_i}) := \sum_{k=0}^{i} (-1)^k D_{j_k}(a) e_{j_1} \wedge \cdots \wedge \hat{e}_{j_k} \wedge \cdots \wedge e_{j_i}.$$

We may rewrite the L-function as follows.

$$\begin{aligned}
L(x_0 f, T)^{(-1)^n} &= \prod_{i=0}^{n+1} det(I - Tq^i\phi_a|B)^{(-1)^i\binom{n+1}{i}} \\
&= \prod_{i=0}^{n+1} det(I - Tq^i\phi_a|B^{\binom{n+1}{i}})^{(-1)^i} \\
&= \prod_{i=0}^{n+1} det(I - Tq^i\phi_a|H_i(K_\bullet(B, D_0, \ldots, D_n)))^{(-1)^i}.
\end{aligned}$$

Now, if f is Δ-regular, then all the homology spaces are trivial except for $i = 0$, in which case

$$H_0(K_\bullet(B, D_0, \ldots, D_n)) = B/\sum_{i=0}^{n} D_i(B)$$

is a free $\mathbb{Z}_q[\pi]$-module of rank $d(\Delta)$. That is the essence of the next two theorems.

Theorem 2.3 (Adolphson-Sperber). If f is Δ-regular, then $L(x_0 f, T)^{(-1)^n}$ is a polynomial of degree $d(\Delta) = n! \operatorname{Vol}(\Delta)$.

Theorem 2.4 (Denef-Loeser). If f is Δ-regular, then $L(x_0 f, T)^{(-1)^n}$ is mixed of weight $\leq n+1$. That is, if

$$L(x_0 f, T)^{(-1)^n} = \prod_{i=1}^{d(\Delta)} (1 - \alpha_i T),$$

then

$$|\alpha_i| = \sqrt{q}^{w_i},\ w_i \in \mathbb{Z} \cap [0, n+1].$$

Let

$$e_j = \#\{1 \leq i \leq d(\Delta) \mid w_i = j\}, \quad 0 \leq j \leq n+1.$$

There exists a very complicated combinatorial formula for e_j.

Example: Let Δ be a simplex and

$$c_0 = 1, c_i = \sum_{\substack{\tau \subset \Delta, \text{face} \\ \dim \tau = i-1}} \operatorname{Vol}(\tau), \quad i \geq 1.$$

Then

$$e_0 = 1, e_j = \sum_{i=0}^{j} (-1)^{j-i} i! \binom{n+1-i}{n+1-j} c_i, \quad j \geq 1.$$

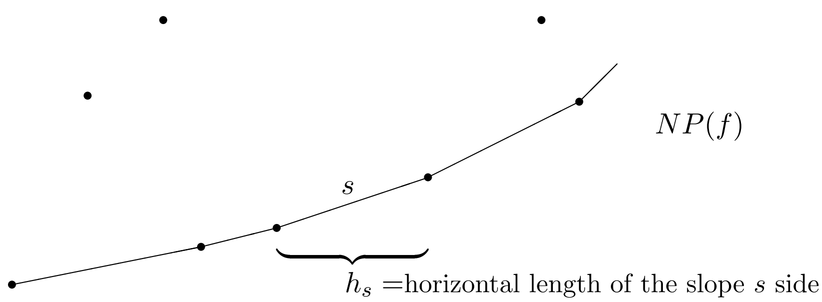
Exercise: $f(\lambda, x) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} - \lambda$, Δ-regular. Compute e_j.

2.7 Newton polygon for $L(x_0 f, T)^{(-1)^n}$

Let f be Δ-regular over $\mathbb{F}_q$. Write

$$L(x_0 f, T)^{(-1)^n} = \sum_{m=0}^{d(\Delta)} A_m T^m, \quad A_0 = 1, A_m \in \mathbb{Z}.$$

Define the q-adic Newton polygon of $L(x_0 f, T)^{(-1)^n}$ to be the lower convex closure in $\mathbb{R}^2$ of the points $(m, \mathrm{ord}_q(A_m))$, $m = 0, 1, \cdots, d(\Delta)$. Denote this polygon by $NP(f)$.



Newton polygon of f

Theorem 2.5. $NP(f)$ has a side of slope s with horizontal length h_s iff there are exactly h_s reciprocal zeros α_i's such that

$$\mathrm{ord}_q(\alpha_i) = s, \quad \text{i.e.}, |\alpha_i| = q^{-s}.$$

Question. For $s \in \mathbb{Q} \cap [0, n+1]$, $h_s = ?$

Theorem 2.6 (Adolphson-Sperber). f is Δ-regular $\Rightarrow NP(f) \geq HP(\Delta)$, with endpoints coincide, where $HP(\Delta)$ is the Hodge polygon of Δ.

An outline of the proof is as follows. See section 2.6 for some relevant notions. We define an operator ϕ_1 on our Banach module $B := S_{\Delta,p} \otimes \mathbb{Z}_q[\pi]$. This induces an operator on the finite dimensional homology space

$$H_0 := H_0(K_\bullet(B, D_0, \ldots, D_n)) = B / \sum_{i=0}^{n} D_i(B)$$

and so may be represented by a matrix if we provide a basis. Choosing a monomial basis $\Gamma_I := \{\pi^{u_0} x^u | u \in I\}$, we may explicitly estimate the p-adic order of the entries of the matrix A_1 representing ϕ_1 to get a (Hodge) filtration: $\phi_1(\Gamma_I) = \Gamma_I A_1$, where

$$A_1 = \begin{pmatrix} M_{00} & M_{01} & M_{02} & \cdots \\ pM_{10} & pM_{11} & pM_{12} & \cdots \\ p^2M_{20} & p^2M_{21} & p^2M_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where M_{ij} is a matrix with $h_\Delta(i)$ rows and $h_\Delta(j)$ columns. Further, the entries of M_{ij} have $ord_p \geq 0$. Relating this to our operator ϕ_a via the relation $\phi_1^a = \phi_a$ and using an argument of Dwork's, we may show the q-adic Newton polygon of $det(I - T\phi_a|H_0)$ lies above the Hodge polygon, which is defined as the lower convex hull of the points

$$\left(\sum_{i=0}^{m} h_\Delta(i), \sum_{i=0}^{m} i \cdot h_\Delta(i)\right)_{m=0,1,\ldots,n}.$$

Definition. If $NP(f) = HP(\Delta)$, then f is called ordinary. In this case, $L(x_0 f, T)^{(-1)^n}$ has exactly $h_\Delta(k)$ reciprocal zeros α_i's such that $\mathrm{ord}_q(\alpha_i) = k$ for all $0 \leq k \leq n$.

2.8 Variation of $NP(f)$ with p

Conjecture: Let $f \in \mathbb{Z}[x_1^\pm, \cdots, x_n^\pm]$ be Δ-regular. Then there exist infinitely many primes p such that $NP(f \bmod p) = HP(\Delta)$ (p is then called ordinary). One further conjectures that the density $\delta(f)$ of ordinary primes exists and is positive.

Example. If $f = x_1 + x_2 + \frac{1}{x_1 x_2} - \lambda$ is Δ-regular and hence defines an elliptic curve over $\mathbb{Q}$, the density $\delta(f)$ is either 1/2 if f has CM (Deuring) or 1 if f has no CM (Serre).

2.9 Variation of $NP(f)$ with f (p fixed)

Let

$$M_p(\Delta)(\overline{\mathbb{F}}_p) = \{f \in \overline{\mathbb{F}}_p[x_1^\pm, \cdots, x_n^\pm] \mid \Delta(f) = \Delta,\ f \text{ is } \Delta\text{-regular}\}.$$

This set is non-empty if $p > d(\Delta)$.

$f \in M_p(\Delta)(\overline{\mathbb{F}}_p) \Rightarrow f \in M_p(\Delta)(\mathbb{F}_q)$ for some q.
$\Rightarrow$ q-adic $NP(f)$ is defined, independent of
the choice of the defining field $\mathbb{F}_q$.

The relatively cohomology is locally free and thus forms an overconvergent σ-module and in fact an overconvergent F-crystal on $M_p(\Delta)$. We obtain

Theorem 2.7 (Grothendieck-Katz). The global minimun

$$GNP(\Delta, p) = \inf_{f \in M_p(\Delta)} NP(f)$$

exists and is precisely attained for all f in a Zariski open dense subset $U_p(\Delta) \hookrightarrow M_p(\Delta)$. This minimun polygon $GNP(\Delta, p)$ is called the generic Newton polygon of the family $M_p(\Delta)$.

Thus, Newton polygon goes up under specialization, that is, for $f \in M_p(\Delta)$,

$$NP(f) \geq GNP(\Delta, p) \geq HP(\Delta).$$

The first equality holds for all $f \in U_p(\Delta)$.

Definition: If $GNP(\Delta, p) = HP(\Delta)$, Δ is called ordinary at p or generically ordinary at p.

Question: Which primes p are ordinary for Δ?

2.10 Generically ordinary primes

Conjecture (Adolphson-Sperber): Δ is ordinary for $p \gg 0$.

Proposition 2.8. Let Δ be minimal (i.e., no lattice points on Δ other than vertices). If $p \equiv 1(\bmod d(\Delta))$, then Δ is ordinary at p

For minimal Δ, $x_0 f$ becomes a diagonal, the L-function can be computed directly using Gauss sums and the slopes can be found using the Stickelberger theorem. This is the local case. Note also for minimal Δ, one has $d(\Delta) = 1$ if $n \leq 2$.

Theorem 2.9 (Wan). 1) If $n \leq 3$, Δ is ordinary for $p > d(\Delta)$.

2) If $n \geq 4$, there exists n-dimensional Δ which is NOT ordinary for all primes p in a certain congruence class.

3) There exists $D^*(\Delta) > 0$ such that Δ is ordinary for $p \equiv 1(\bmod D^*(\Delta))$.

Part 1) and part 3) follow from the collapsing decomposition (to be explained in the lectures) and a finer form of the above local proposition.

Conjecture. There is a positive integer $\mu(\Delta)$ such that the set of almost all (except for finitely many) ordinary primes for Δ consists of the primes in certain congruence classes modulo $\mu(\Delta)$.

2.11 Generically ordinary Calabi-Yau hypersurfaces

Theorem 2.10 (Wan). Let Δ be reflexive.

1) If $n = \dim(\Delta) \leq 4$, then Δ is ordinary for $p > d(\Delta)$.

2) If Δ is Fano, then Δ is always ordinary for every p.

Part 2) follows from the star decomposition theorem. The case $n = 4$ of Part 1) follows from a combination of the star decomposition and the collapsing decomposition (to be explained in the lectures).

Questions:

1) For reflexive Δ with $n = \dim(\Delta) \geq 5$, is Δ ordinary for $p > d(\Delta)$?

2) If Δ is reflexive and ordinary at $p > d(\Delta)$, is Δ^* ordinary at p?

(already yes if $n \leq 4$ or Δ is Fano)

2.12 Basic example

Take

$$f(\lambda, x) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} - \lambda.$$

Then

$$\begin{aligned} \Delta =& \Delta(f) = \langle e_1, e_2, \cdots, e_n, -(e_1 + e_2 + \cdots + e_n) \rangle, \\ \Delta^* =& \langle (n, -1, \cdots, -1), \cdots, (-1, -1, \cdots, n), (-1, -1, \cdots, -1) \rangle. \end{aligned}$$

Theorem 2.11. Both Δ and Δ^* are ordinary for all primes p.

Theorem 2.12. 1) Let $f(\lambda, x)$ be Δ-regular over $\mathbb{F}_q$. Then

$$L(x_0 f, T)^{(-1)^n} = \prod_{i=0}^{n} (1 - \alpha_i(\lambda) T), \quad d(\Delta) = n + 1.$$

2) $\alpha_0(\lambda) = 1$, $|\alpha_i(\lambda)| = \sqrt{q}^{n+1}$.

3) Δ is ordinary at p. That is, except for finitely many λ, we have $\mathrm{ord}_q(\alpha_i(\lambda)) = i, 1 \leq i \leq n$.

Proof of 2). Since endpoints for $NP(f)$ and $HP(\Delta)$ coincide, we have

$$\mathrm{ord}(\alpha_0 \alpha_1 \cdots \alpha_n) = \mathrm{ord}(\alpha_1 \cdots \alpha_n) = \sum_{k=0}^{n} k h_\Delta(k) = \frac{n(n+1)}{2}.$$

By Denef-Loeser (Theorem 2.2), $|\alpha_i| \leq q^{\frac{n+1}{2}}$. Now $\alpha_1 \cdots \alpha_n \in \mathbb{Z}$ implies that $(\alpha_1 \cdots \alpha_n)^2 = \alpha_1 \cdots \alpha_n \overline{\alpha}_1 \cdots \overline{\alpha}_n \leq q^{n(n+1)}$.

But $\mathrm{ord}(\alpha_1 \cdots \alpha_n)^2 = n(n+1)$, then $(\alpha_1 \cdots \alpha_n)^2 = q^{n(n+1)}$. Hence $|\alpha_i| = q^{\frac{n+1}{2}}$ for all $1 \leq i \leq n$.

Let $g(\lambda, x) = x_0^{n+1} + \cdots + x_n^{n+1} - \lambda x_0 x_1 \cdots x_n$. For almost all λ, it defines a smooth projective hypersurface in $\mathbb{P}^n$. Then

$$Z(g(\lambda, x), T) = \frac{P(T)^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)},$$

where $P(T)$ is a polynomial of degree $\frac{n(n^n-(-1)^n)}{n+1}$, pure of weight $n-1$. The Newton polygon of $P(T) \geq HP$ (Dwork).

Question: Is this family $g(\lambda, x)$ of projective hypersurfaces in $\mathbb{P}^n$ generically ordinary for all $p > n+1$? (yes for $n \leq 3$.)

2.13 Zeta functions of affine toric hypersurfaces

Let $f \in \mathbb{F}_q[x_1^{\pm}, \cdots, x_n^{\pm}], \Delta = \Delta(f), f$ is Δ-regular. Then trivially $T = 1$ is a root of $L(x_0 f, T)^{(-1)^n}$. And

$$\frac{L(x_0 f, T)^{(-1)^n}}{1-T} = P(f, qT)$$

is a polynomial in $1 + T\mathbb{Z}[T]$ of degree $d(\Delta) - 1$ (with slope ≥ 1), where $P(f, T)$ is a polynomial in $\mathbb{Z}[T]$. We have

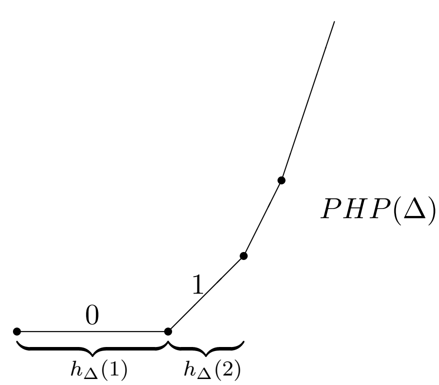
$$\begin{aligned} Z(U_f, qT) &= \prod_{i=0}^{n} (1 - q^i T)^{(-1)^{n-i-1}\binom{n}{i}} L(x_0 f, T) \\ &= \prod_{i=1}^{n} (1 - q^i T)^{(-1)^{n-i-1}\binom{n}{i}} \left(\frac{L(x_0 f, T)^{(-1)^n}}{1-T}\right)^{(-1)^n} \\ &= \prod_{i=1}^{n} (1 - q^i T)^{(-1)^{n-i-1}\binom{n}{i}} P(f, qT)^{(-1)^n}, \\ Z(U_f, T) &= \prod_{i=0}^{n-1} (1 - q^i T)^{(-1)^{n-i}\binom{n}{i+1}} P(f, T)^{(-1)^n}, \end{aligned}$$

where

$$P(f, T) = \prod_{i=0}^{d(\Delta)-2} (1 - \beta_i T)$$

and the β_i's are algebraic numbers.

Definition 2.13. The primitive Hodge polygon $PHP(\Delta)$ is the polygon in $\mathbb{R}^2$ with vertices $(0, 0)$ and $(\sum_{k=0}^{m} h_\Delta(k), \sum_{k=0}^{m} (k-1) h_\Delta(k)), 1 \leq m \leq n$.



Primitive Hodge polygon of Δ

$$P(f,T) = \det(I - Frob_q T \mid \mathrm{PH}_c^{n-1}(U_f \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_\ell))$$
$\Longrightarrow$ All results on $L(x_0 f, T)^{(-1)^n}$ carry over to $P(f,T)$.

Corollary 2.14. If f is Δ-regular over $\mathbb{F}_q$, then

$$\#U_f(\mathbb{F}_{q^k}) = \frac{(q^k-1)^n + (-1)^{n+1}}{q^k} + (-1)^{n+1}(\beta_0^k + \beta_1^k + \cdots + \beta_{d(\Delta)-2}^k),$$

where $|\beta_i| \leq q^{\frac{n-1}{2}}$.

3 p-adic Variation

3.1 p-adic Analytic formula for the Frobenius matrix

Let $f(\overline{\lambda}, x)$ be the universal family of $f \in M_p(\Delta)$. For $f(\overline{\lambda}, x) \in M_p(\Delta)(\mathbb{F}_q)$,

$$P(f(\overline{\lambda}, x), T) = \det(I - Frob_q T \mid \mathrm{PH}_c^{n-1}) = \det(I - F(\lambda)T).$$

Here $F(\lambda$ is a matrix of size $(d(\Delta)-1) \times (d(\Delta)-1)$. Is there any p-adic analytic formula for $F(\lambda)$? Since the relative cohomology forms a locally free overconvergent σ-module, one obtains

Theorem 3.1. Zariski locally on $M_p(\Delta)$, there exists an overconvergent matrix $A(\lambda)$ of size $(d(\Delta)-1) \times (d(\Delta)-1)$ of the form

$$A(\lambda) = \begin{pmatrix} A_{00}(\lambda) & A_{00}(\lambda) & \cdots \\ pA_{10}(\lambda) & pA_{11}(\lambda) & \cdots \\ \vdots & \vdots & \\ p^{n-1}A_{n-1,0}(\lambda) & p^{n-1}A_{n-1,1}(\lambda) & \cdots \end{pmatrix},$$

where $A_{ij}(\lambda)$ has $h_\Delta(i+1)$ rows and $h_\Delta(j+1)$ columns, whose entries are overconvergent functions on the lifting of $M_p(\Delta)$ with norm ≤ 1, satisfies the following property:

if $f(\overline{\lambda}, x) \in M_p(\Delta)(\mathbb{F}_{p^a})$ and $\lambda = \mathrm{Teich}(\overline{\lambda})$, then one can take

$$F(\lambda) = A(\lambda^{p^{a-1}}) \cdots A(\lambda^p)A(\lambda).$$

That is, $P(f(\overline{\lambda}, x), T) = \det(I - A(\lambda^{p^{a-1}}) \cdots A(\lambda^p)A(\lambda)T)$.

3.2 Deformation theory and Picard-Fuch equation

Let $p > 2$. Since the relatively cohomology forms an overconvergent F-crystal whose underlying differential equation is the Picard-Fuch equation, we deduce that the matrix $A(\lambda)$ as above can be expressed in terms of a fundamental solution matrix $C(\lambda)$ of the Picard-Fuch equation:

$$A(\lambda) = C(\lambda^p)^{-1}A(\lambda_0)C(\lambda),$$

where λ_0 is a regular point.

Remark: $C(\lambda)$ is NOT analytic on the closed unit disk near λ_0.

Example: Let

$$\pi^{p-1} = -p, \lambda^{p^a} = \lambda, \theta(\lambda) = \exp(-\pi\lambda^p)\cdot\exp(\pi\lambda) \rightsquigarrow A(\lambda),$$

$$\Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}(\overline{\lambda}) = \theta(\lambda^{p^{a-1}})\cdots\theta(\lambda^p)\theta(\lambda).$$

If $P(f(\overline{\lambda}, x), T) = \prod_{i=0}^{d(\Delta)-2}(1-\alpha_i(\lambda)T) \in \mathbb{Z}[T]$ has $h_\Delta(k+1)$ reciprocal roots with slope k, $k = 0, 1, \cdots, n-1$, then

$$\begin{aligned} P(f(\overline{\lambda}, x), T) = &\prod_{i=0}^{n-1} P_i(\lambda, T), \quad P_i(\lambda, T) \in \mathbb{Z}_p[T], \\ \deg P_i(\lambda, T) = &h_\Delta(i+1) \\ &i = \text{slope of } P_i(\lambda, T). \end{aligned}$$

Question: Any p-adic analytic formula for $P_i(\lambda, T)$?

3.3 Hodge-Newton decomposition and unit root formula

Let Δ be ordinary at $p > 2$ and $H_p(\Delta)$ be the ordinary locus of $M_p(\Delta)$. One wishes to find a new basis such that the new matrix

$$\begin{pmatrix} I_{00} & -E_{01}(\lambda^p) \\ 0 & I_1 \end{pmatrix} A(\lambda) \begin{pmatrix} I_{00} & E_{01}(\lambda) \\ 0 & I_1 \end{pmatrix} = \begin{pmatrix} B_{00}(\lambda) & * \\ 0 & pA'(\lambda) \end{pmatrix}.$$

This defines a p-adic contraction map and thus it has a unique solution matrix $E_{01}(\lambda)$ which is a convergent matrix of $h_\Delta(0)$ rows and $\sum_{i=1}^n h_\Delta(i)$ columns. By induction, one then shows that there exists a convergent matrix $D(\lambda)$ on the lifting of $H_p(\Delta)$ such that

$$D(\lambda^p)^{-1}A(\lambda)D(\lambda) = \begin{pmatrix} B_{00}(\lambda) & * & * \\ 0 & pB_{11}(\lambda) & * \\ 0 & 0 & \ddots \end{pmatrix}.$$

Then

$$P(f(\overline{\lambda}, x), T) = \prod_{i=0}^{n-1} \det\left(I - p^{ai}B_{ii}(\lambda^{p^{a-1}})\cdots B_{ii}(\lambda^p)B_{ii}(\lambda)T\right),$$

where $f(\overline{\lambda}, x) \in H_p(\Delta)(\mathbb{F}_{p^a})$, $B_{ii}(\lambda)$ is convergent (not overconvergent) on the closed unit disk,

$$B_{ii}(\lambda) = C_{ii}(\lambda^p)^{-1}B_{ii}(0)C_{ii}(\lambda),$$

where $C_{ii}(\lambda)$ is a fundamental solution matrix of a piece of the Picard-Fuch equation. The p-adic analytic formula for $P_i(\lambda, T)$ is then

$$P_i(\lambda, T) = \det\left(I - p^{ai} B_{ii}(\lambda^{p^{a-1}}) \cdots B_{ii}(\lambda^p) B_{ii}(\lambda) T\right).$$

3.4 Unit root L-function and p-adic Galois representation

Let Δ be ordinary at p. As above,

$$A(\lambda) \sim \begin{pmatrix} B_{00}(\lambda) & * & * & * \\ 0 & pB_{11}(\lambda) & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & p^{n-1} B_{n-1,n-1}(\lambda) \end{pmatrix}.$$

Each $B_{ii}(\lambda)$ is invertible on the lifting of $H_p(\Delta)$ and hence it defined a unit root F-crystal on $H_p(\Delta)$. Alternatively, we have

Theorem 3.2 (Katz). Each B_{ii} defines a continuous p-adic representation

$$\rho_i : \pi_1^{\text{arith}}(H_p(\Delta)/\mathbb{F}_p) \longrightarrow GL_{h_\Delta(i+1)}(\mathbb{Z}_p),$$

such that

$$\rho_i(\text{Frob}_\lambda) = B_{ii}(\lambda^{p^{a-1}}) \cdots B_{ii}(\lambda^p) B_{ii}(\lambda).$$

It is clear that the L-function

$$L(\rho_i, T) = \prod_{\substack{\bar{\lambda} \in H_p(\Delta) \\ \text{closed point}}} \frac{1}{\det\left(I - T^{\deg(\lambda)} \rho_i(\text{Frob}_\lambda)\right)} \in 1 + T(\overline{\mathbb{Q}} \cap \mathbb{Z}_p)[[T]]$$

is analytic in $|T|_p < 1$.

3.5 Dwork's unit root conjecture

Theorem 3.3 (Wan). $L(\rho_i, T)$ is p-adic meromorphic everywhere.

Let $\rho = \rho_i$. Write the p-adic Weierstrass factorization

$$L(\rho_i, T) = \frac{\prod_{j=1}^\infty (1 - z_j^{(1)} T)}{\prod_{j=1}^\infty (1 - z_j^{(2)} T)}, \quad z_j^{(1)} \to 0, z_j^{(2)} \to 0.$$

<u>**Question**</u>: Let $K_p = \mathbb{Q}_p(z_j^{(1)}, z_j^{(2)} | 1 \le j < \infty)$. Is $[K_p : \mathbb{Q}_p] < \infty$? ($p$-adic RH for $L(\rho_i, T)$).

Definition 3.4. Let

$$r_\rho^+ = \limsup_{x \to \infty} \frac{\log(1 + \#\{i \mid \text{ord}_q(z_i^{(1)}) \le x\} + \#\{j \mid \text{ord}_q(z_j^{(2)}) \le x\})}{\log x}$$

This is called the order of the p-adic meromorphic function $L(\rho, T)$. It measures the size of $L(\rho, T)$. Clearly, we have $0 \leq r_\rho^+ \leq +\infty$.

Question: $r_\rho^+ < +\infty$?

Theorem 3.5 (Wan). If $\mathrm{rank}(\rho) = 1 \Rightarrow r_\rho^+ < +\infty$.

(true for the family $x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} - \lambda$)

Definition 3.6. Let

$$r_\rho^- = \limsup_{x \to \infty} \frac{\log(1 + |\#\{i \mid \mathrm{ord}_q(z_i^{(1)}) \leq x\} - \#\{j \mid \mathrm{ord}_q(z_j^{(2)}) \leq x\}|)}{\log x}$$

Clearly, $0 \leq r_\rho^- \leq r_\rho^+ \leq +\infty$.

Question: $r_\rho^- < +\infty$? (yes if $\mathrm{rank}(\rho) = 1$)

3.6 p-adic Monodromy group

Let Δ be ordinary at p. Let

$$\rho = \rho_i : \pi_1^{\mathrm{arith}}(H_p(\Delta)/\mathbb{F}_p) \longrightarrow GL_{h_\Delta(i+1)}(\mathbb{Z}_p).$$

Then $G_p(\Delta, i) = \rho_i(\pi_1^{\mathrm{arith}})$ is a p-adic Lie-group.

Question: $G_p(\Delta, i) =$?

Example (Igusa). For the elliptic family $x_1 + x_2 + \frac{1}{x_1 x_2} - \lambda$, one has

$$G_p(\Delta, 0) = G_p(\Delta, 1) = \mathbb{Z}_p^* = GL_1(\mathbb{Z}_p).$$