Cohomology, periods and the Hodge structure of toric hypersurfaces

(Course description)

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Abstract

The aim of this course is to study the cohomology groups $H^*(Z_f)$ of non-degenerate affine toric hypersurfaces $Z_f \subset (\mathbf{C}^*)^d$. Some properties of the cohomology groups can be described in terms of the Newton polytope of the equation f. We relate the periods of Z_f to the GKZ-hypergeometric functions and give applications in physics and number theory.

1 Introduction

Let $M \cong \mathbf{Z}^d$ be a free abelian group of rank d. We identify M with the group of characters of the d-dimensional torus $\mathbf{T}_d \cong (\mathbf{C}^*)^d$

$$\mathbf{T}_d = \operatorname{Spec} \mathbf{C}[M] \cong \operatorname{Spec} \mathbf{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}].$$

Let $\Delta \subset M \otimes \mathbf{R}$ be a d-dimensional convex polytope such that all vertices of Δ belong to the lattice M. We choose a finite subset $A \subset \Delta \cap M$ which contains all vertices of Δ and consider a Laurent polynomial

$$f = f(X_1, \ldots, X_d) = \sum_{m \in A} a_m X^m,$$

where a_m $(m \in A)$ are sufficiently general complex numbers.

We will be interested in cohomology groups $H^i(Z_f, \mathbf{Z})$ of the affine hypersurface Z_f in $\mathbf{T}_d$ defined by the equation $f = 0$. Since Z_f is affine, one has $H^i(Z_f, \mathbf{Z}) = 0$ for $i \geq d$. By the Lefschetz-type theorem, one obtains the isomorphisms

$$H^i(Z_f, \mathbf{Z}) \cong H^i(\mathbf{T}_d, \mathbf{Z}) = \Lambda^i M, \ \ i < d - 1.$$

Therefore the groups $H^{d-1}(Z_f, \mathbf{Z})$ and $H^{d-1}(Z_f, \mathbf{C})$ are the only interesting objects for our study.

The group $H^{d-1}(Z_f, \mathbf{C})$ has a mixed Hodge structure which can be characterized by Hodge-Deligne numbers [7]. On the other hand, the periods, i.e., integrals of $(d-1)$-differential forms on Z_f over (d-1)-dimensional cycles satisfy a system of differential equations of Picard-Fuchs type. These differential equations are important for applications in physics [4] and they have p-adic analogs [8] which are related to the Zeta-function of Z_f over a finite field.

2 Course content

1. The toric compactification of $\mathbf{C}^d$ with respect to a lattice polytope Δ. The nondegeneracy condition for hypersurfaces $Z_f \subset \mathbf{C}^d$. The Euler number of Z_f. The number of critical points of f in $\mathbf{C}^d$. The Lefschetz-type theorem for Z_f.

2. De Rham cohomology of a nondegenerate hypersurface $Z_f \subset \mathbf{C}^d$. Logarithmic de Rham complex. Principal A-determinant of f in the sense of Gelfand-Kapranov-Zelevinsky [11]. Jacobian ring R_f and its canonical module [2]. Cohomology with compact supports. Duality and toric residues. Hodge-Deligne numbers of Z_f [5].

3. Generalized hypergeometric differential system of Gelfand-Kapranov-Zelevinsky [10]. The dimension of the solution space of GKZ-system. Coherent triangulations of the Newton polytope and a basis of the solution space. Generalized GKZ-hypergeometric functions as periods of hypersurfaces Z_f.

4. The secondary polytope $\mathrm{Sec}(\Delta)$ as the Newton polytope of the pricipal A-determinant of f. The asymptotics of complex and real hypersurfaces corresponding to vertices of $\mathrm{Sec}(\Delta)$. The monodromy of 1-parameter familites. The method of Viro and methods of tropical geometry [13].

5. Applications in physics and number theory. The toric mirror symmetry [3]. Monomial-divisor mirror corespondence [1]. The Seiberg duality. P-adic versions of GKZ-hypergeometric functions and period. Affine toric Fermat-type hypersurfaces.

3 Student project

First interesting examples for study are families of affine algebraic curves $Z_f \subset \mathbf{T}_2$ defined by a 2-dimensional polytope (polygone) $\Delta \subset M \otimes \mathbf{R}$. If n is the number of lattice points on the boundary of Δ and g is the number of interior lattice points in Δ, then Z_f can be seen as a Riemann surface $\overline{Z_f}$ of genus g minus n points. Periods of $\overline{Z_f}$ are classical objects of algebraic geometry [6].

4 Prerequisites

It is recommended to have some background on algebraic geometry (see e.g. the book of Griffiths and Harris [12]) and toric geometry (see e.g. the book of Fulton [9]).

References

[1] P.S. Aspinwall, B.R. Greene, and D.R. Morrison, . *The monomial-divisor mirror map*, Int. Math. Res. Not., No.12, (1993), 319-337.

[2] V.V. Batyrev *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*. Duke Math. J. 69 (1993), no. 2, 349–409.

[3] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom., **3** (1994), 493-535.

[4] V.V. Batyrev and D. van Straten, *Generalized Hypergeometric Functions and Rational Curves on Calabi-Yau complete intersections in Toric Varieties*, Commun. Math. Phys. **168** (1995), 493-533.

[5] V.V. Batyrev, L.A. Borisov, *Mirror duality and string-theoretic Hodge numbers*, Invent. Math., **126** (1996), p. 183-203.

[6] H. Clemens, *A scrapbook of complex curve theory*. Second edition. Graduate Studies in Mathematics, 55. AMS, Providence, RI, 2003.

[7] Danilov, V. I.; Khovanski, A. G. *Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), no. 5, 925–945.

[8] B. Dwork, *Lectures on p-adic differential equations* Grundlehren der Mathematischen Wissenschaften, 253. Springer-Verlag, New York-Berlin, 1982

[9] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[10] Gelfand, I. M.; Zelevinski, A. V.; Kapranov, M. M. *Hypergeometric functions and toric varieties* Funct. Anal. Appl. 23 (1989), no. 2, 94–106

[11] Gelfand, I. M.; Zelevinski, A. V.; Kapranov, M. M. *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994

[12] Ph. Griffiths, J. Harris, *Principles of algebraic geometry* John Wiley & Sons, Inc., New York, 1994.

[13] G. Mikhalkin, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, math.GT/0205011, to appear in Topology.

[14] M. Oka, *On the topology of full nondegenerate complete intersection variety*, Nagoya Math. J. 121 (1991), 137–148.