K Arithmetic/Geometric

finite extension

$k(t_1, \ldots, t_d)$

purely transcendental

$k =$ Prime Field/Algebraically Closed Field

Case 3 $k = \mathbb{Q}$

Fact Let $K/_{\mathbb{Q}}$ be an arithmetic function field then:

1. $\mathbf{td}(K/_{\mathbb{Q}}) = d \iff$

$$\begin{cases} \forall\ t_1, \ldots t_{d+2} \in K q_{(t_1,\ldots,t_{d+2})} \text{ is universal over } K(\sqrt{-1}) \\ \textbf{and} \\ \exists\ t_1, \ldots, t_{d+2} \in K \text{ such that } q_{(t_1,\ldots,t_{d+2})} \\ \qquad\qquad \text{does not represent 0 over } K(\sqrt{-1}) \end{cases}$$

   i.e. by a conjunction of a $\forall\exists$ and $\exists\forall\exists$ sentence

2. Let $(t_1, \ldots, t_d)$ be a transcendence base (TB) then $\exists$ "many" $a_1, \ldots, a_d, a, b \in \mathbb{N}$ such that $q_{(t_1-a_1,\ldots,t_d-a_d)}$ does not represent 0 over $K(\sqrt{-1})$

Proof Idea Cohomology of finitely generated fields and Milnor conjecture

Theorem A K, L arithmetic function fields, $K \equiv L$, then

1. $\exists$ separable field embeddings $K \hookrightarrow L$ and $K \hookrightarrow K$
2. If K is of general type $\implies K \cong L$

Proof (Case **Char** $K = 0$)

- Choose $t_1, \dots, t_d, a, b$ such that $q_{(t_1,\dots,t_d,a,b)}$ is universal but does not represent 0 over $K(\sqrt{-1})$ where $d = \mathbf{td}(K/_{\mathbb{Q}})$
- Choose $x \in K$ such that $K = \mathbb{Q}(t_1, \dots, t_d, x)$
- Let $P(T_1, \dots, T_d, X)$ be the defining polynomial relation over $\mathbb{Q}$

- Consider the sentence Ψ:

$$\begin{cases} \exists x_1, \ldots, x_d, x \text{ such that } P(x_1, \ldots, x_d, x) = 0 \\ \textbf{and} \\ q_{(x_1,\ldots,x_d,a,b)} \text{ is universal but does not represent0 over } K(\sqrt{-1}) \end{cases}$$

- $K \models \Psi \implies L \models \Psi$
- Thus $\exists\, u_1, \ldots, u_d \in L$, $y \in L$ such that $P(u_1, \ldots, u_d, y) = 0$ and $q_{(u_1,\ldots,q_d,a,b)}$ is universal but does not represent 0 over $L(\sqrt{-1})$
- Since $\mathbf{td}(L/_{\mathbb{Q}}) = d$ (Why?) it follows $U - 1, \ldots, u_d$ is a TB of $L/_{\mathbb{Q}}$
- Since $P(T_1, \ldots, T_d, x)$ is irreducible over $\mathbb{Q} \implies$ $(t_1, \ldots, t_d, x) \longmapsto (u_1, \ldots, u_d, y)$ defines a field embedding $K \hookrightarrow L$

Suppose K is of general type

"Introduction to general type"

- **td** $= 1$

$$g = 0 \qquad \text{———}$$

Not of general type

$$g = 1 \qquad \text{———}$$

$$g > 1$$

- **Theorem** K of general type, $\phi : K \underset{k}{\hookrightarrow} K$ embedding $\implies$ ϕ is a k-isomorphism

- $j : K \overset{i}{\hookrightarrow} L \overset{i'}{\hookrightarrow} K$ $\quad$ $\mathbb{Q}$-embeddings
- Let $k = K \cap \overline{\mathbb{Q}} = k^{\mathbf{abs}}$ be the constants
- Then $k/_{\mathbb{Q}}$ is finite
- j maps k into itself thus $j|_k : k \longrightarrow k$ is an isomorphism (Why?)
- Therefore $\exists\, n$ such that $j^{\circ n} : K \longrightarrow K$ is a k-embedding
- Therefore $j^{\circ n}$ is a ????k-isomorphism
- j is a field isomorphism
- i, i' are isomorphisms

Questions/Comments

1. What about case $\mathbf{td}(L/_{\mathbb{Q}}) = 1$, $K = \mathcal{K}(X)$ with $g_X = 0$ or 1?

2. What about elementary equivalence and Galois descent?

    - K, L, arithmetic function fields
    - $K \equiv L$
    - k prime field
    - Let $k'/_k$ be (big enough) Galois extension such that $K \cdot k' \cong L \cdot k'$ as a field
    - Show $K \cong L$

Case $K/_k$ geometric function field

k algebraically closed field

$K/_k$, $L/_k$ function fields with $K \equiv L$

First Approximation

- Choose $(t_1, \dots, t_d)$ separable TB of $K/_k$
- Choose $x \in K$ such that $K = k(t_1, \dots, t_d, x)$ and $P(T_1, \dots, T_d, X) \in K[T_1, \dots, T_d, X]$ irreducible defining relation of x
- Let $R = k[\alpha_1, \dots, \alpha_n]$ be sufficiently large of finite type over k over which "$K/_k$ is defined"

  ($k_o = \mathbf{Quot}(R)$ $\quad K_o = k_o(t_1, \dots, t_d, x)$ then $K = K_o k \cdots$

- Set $R = k[z_1, \ldots, z_n]/_{(f_1, \ldots, f_m)}$
- Ψ: $\exists\, \alpha_1, \ldots, \alpha_n$, $\exists\, x_1, \ldots, x_d$, $\exists x$ such that:

$$f_1(\alpha_1, \ldots, \alpha_n) = \cdots = f_m(\alpha_1, \ldots, \alpha_n) = 0$$

and $P \longleftrightarrow (\alpha_1, \ldots, \alpha_n)$ **and**

and $P(x_1, \ldots, x_d, x) = 0$

and

$$\begin{cases} q^p_{(x_1, \ldots, x_d)} \text{ universal but does not represent 0 if } k = \mathbb{F}_p \\ q_{(x_1, \ldots, x_d)} \text{ universal but does not represent 0 if } k = \mathbb{Q} \end{cases}$$

- Interpret Ψ in L:

$$\exists\, b_1, \ldots, b_n, u_1, \ldots, u_d, y \in L \text{ such that } \cdots$$

- Unfortunately $(u_1, \ldots, u_b)$ is a separable TB of $L/_k$

Want a field embedding $K \hookrightarrow L$ via Ψ

- **Problem 1** $b_1, \ldots, b_m$ are possibly not in k
- **Problem 2**
  - Suppose by a supplementary sentence Ψ_1 one can impose $b_1, \ldots, b_n$ are in k
  - Then define

$$
\begin{aligned}
p : R &\longrightarrow k \\
(\alpha_1, \ldots, \alpha_n) &\longmapsto (b_1, \ldots, b_n)
\end{aligned}
$$

  - The let $\mathcal{P} = \mathbf{Ker}(p)$
  - $\mathfrak{X} = \mathbf{Spec}\, R[x_1, \ldots, x_m, x]/_{\mathcal{P}} \longrightarrow \mathbf{Spec}\, R$
  - Then $X_{\mathcal{P}} = \mathfrak{X} \times_R \mathcal{K}(p)$

- Finally we get a function field embedding:

$$\begin{array}{ccc} K_{\mathcal{P}} = \mathcal{K}(X_g) & \hookrightarrow & L \\ \uparrow & & \uparrow \\ \mathcal{K}(\mathcal{P}) & \underset{\mathcal{P}_1}{\hookrightarrow} & k \end{array}$$

- Is p always injective?

- **Problem 1** YES k is definable in K
- **Problem 2 ? ? ?**
- **Question 1** What about definability of k in K if K is real or p-adically closed
- **Question 2** What about Problem **2**?