Elementary Equivalence $\prec\!\!\longrightarrow\!\!\succ$ Isomorphism

Introduction Motivation

- "Classifying Question"

  $K = \mathbb{C}(x_1, \ldots, x_n) \rightsquigarrow \mathbf{td}(K/\mathbb{C})$

- Elementary Equivalence of Fields

  - $\mathfrak{L}$ $\quad +, -, \cdot, 1, 0$ $\quad$ Axioms of fields
  - $\mathcal{A} = \{$ all sentence in $\mathfrak{L}$ $\}$
  - $\mathbf{Th}(K) = \{\phi \in \mathcal{A} \mid K \models \phi\}$
  - $\forall \vec{x_1} \exists \vec{x_2} \ldots \overline{P}(\vec{x_1}, \ldots) = 0$

- **Fact**

$$K \cong L \quad \Longrightarrow \quad \mathbf{Th}(K) = \mathbf{Th}(L)$$
$$\not\Longleftarrow$$

- **Definition** (elementary equivalent) $K \equiv L \overset{\text{def}}{\longleftrightarrow} \mathbf{Th}(K) = \mathbf{Th}(L)$

Facts

1. $\mathbf{Char}(K)$ is encoded in $\mathbf{Th}(K)$
2. K algebraically closed is encoded in $\mathbf{Th}(K)$
3. K is real closed is encoded in $\mathbf{Th}(K)$
   - $\mathbf{v} : K^\times \longrightarrow \Gamma_v \qquad \Gamma_v$ divisible
   - $k_v \subseteq \mathbb{R}$ is relatively algebraically closed
   - $(K, \mathbf{v})$ henselian
4. K p-adically closed is encoded in $\mathbf{Th}(K)$
5. Geometric interpretation of $\equiv$
   - Every $\phi \in \mathcal{A} \longleftrightarrow S_\phi \subseteq \mathbb{A}^N_{\mathbb{Z}}$
   - $K \equiv L \iff$
     $(\forall \phi$ one has $S_\phi(K) \neq \emptyset \iff S_\phi(L) \neq \emptyset)$

7. Relation of $\equiv$ with $\cong$

- Ultrapowers: K, I index set, $\mathcal{D}$ ultrafilter on I,
  $K^* := K^I/_{\mathcal{D}} = K^I/_{\mathcal{M}_{\mathcal{D}}}$
- $K \equiv L \iff \exists K^* = K^I/_{\mathcal{D}}$, $L^* = K^J/_{\mathcal{E}}$ such that $K^* \cong L^*$
- **Remark** $K \equiv K^*$

<u>Theorem</u> (Classification up to $\equiv$)

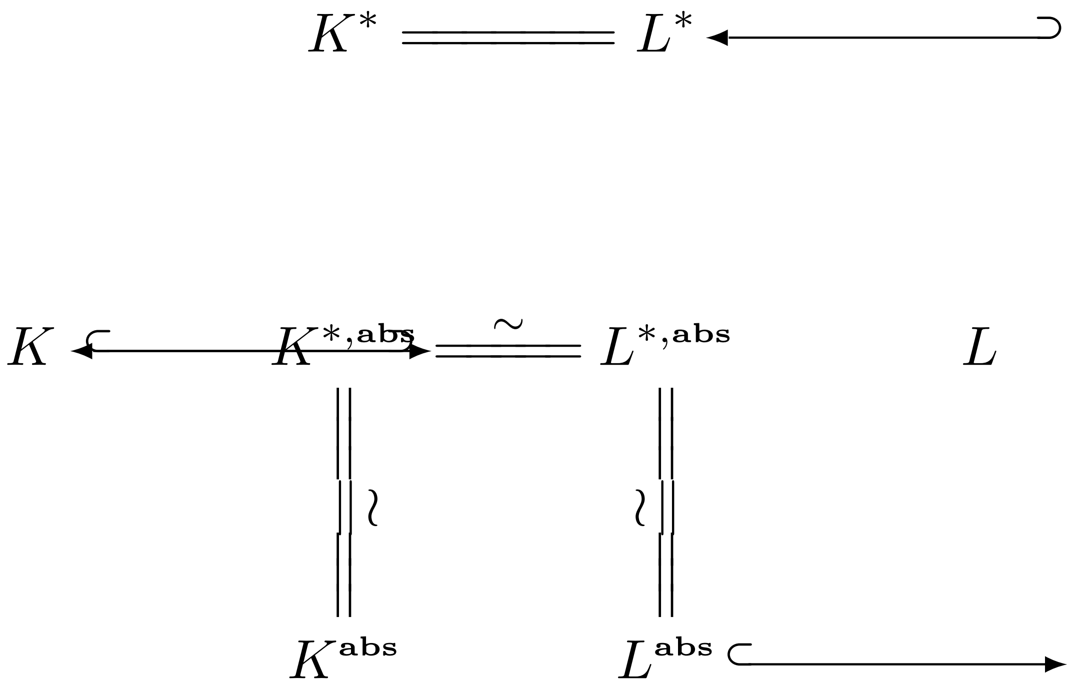
1. $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}}_p$ (all p) are representatives for all algebraically closed fields
2. $\mathbb{R}^{\mathbf{abs}} := \mathbb{R} \cap \overline{\mathbb{Q}}$ is real closed and all real closed fields are elementary equivalent
3. $\mathbb{Q}_p^{\mathbf{abs}} := \mathbb{Q}_p \cap \overline{\mathbb{Q}}$ is p-adically closed and all p-adically closed fields are elementary equivalent

Comment Generalized p-adically closed fields:

$\mathbb{K}/\mathbb{Q}_p$ finite, $\qquad d := [\mathbb{K} : \mathbb{Q}_p], \qquad \mathbb{K}^{\mathbf{abs}} := K \cap \overline{\mathbb{Q}}$

Consequence if $K \equiv L \implies K^{\mathbf{abs}} \cong L^{\mathbf{abs}}$

Proof



The Other Extreme

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prime field $\longleftrightarrow$ arithmetic situation

fields of finite type over ———

alg. cl. field $\longleftrightarrow$ geometric situation

- **Comment**
  - $\mathbb{Q}(t) = \mathcal{K}(\mathbb{P}^1_{\mathbb{Q}})$
  - $\mathbb{Q}(u, v)$
  - $\mathcal{F} = (t_1, \ldots, t_d)$ is a transcendence basis (TB) over k if
    $\forall P(x_1, \ldots, x_d) \neq 0 \in k[x_1, \ldots, x_d] \implies P(t_1, \ldots, t_d) \neq 0$

Arithmetic:

$$\begin{array}{c} K \\ \Big| \ \text{finite} \\ k(t_1, \ldots, t_d) \\ \Big| \ \text{purely transcendental} \\ k \end{array}$$

Geometric:

$$\begin{array}{c} K \\ \Big| \ \text{finite} \\ k(t_1, \ldots, t_d) \\ \Big| \ \text{purely transcendental} \\ k \end{array}$$

$k = \mathbb{F}_p$ or $\mathbb{Q}$

- **Problem** (Elementary Equivalence $\prec\!\!-\!\!\succ$ Isomorphism)

  Describe a set of representatives for the elementary equivalence classes of function fields (arithmetic/geometric situation)

- **Hope** K, L such fields $\iff K \cong L$

(Sabbagh ’85)

- **Theorem A** (Arithmetic Case) Let K and L be arithmetic function fields. Suppose $K \equiv L$ then $\exists$ field embeddings $K \hookrightarrow L$ and $L \hookrightarrow K$. In particular, if K is of general type, then $K \cong L$

- **Theorem B** (Geometric Situation) Let K and L be geometric function fields. Suppose $K \equiv L$ then

  1. $\mathbf{td}(K/k) = \mathbf{td}(L, \lambda) \qquad k \equiv \lambda$
  2. if K is of general type $K \cong L$ provided that $k \cong \lambda$

Comments

- Durét, Pierce: Geometric case under hypothesis $\mathbf{td}(K/k) = 1$ goes beyond Theorem **B**: elliptic curves of non-CM type

- $K^* \cong L^* \qquad \mathcal{C}_L \longrightarrow \mathcal{C}_K \qquad K \hookrightarrow L$