H10/ subrings of $\mathbb{Q}$

- $\mathcal{P} = \{2, 3, 5, \dots\}$
-

$$\{\text{subsets of } \mathcal{P}\} \longleftrightarrow \{\text{subrings of } \mathbb{Q}\}$$

$$S \longmapsto \mathbb{Z}[S^{-1}]$$

$$\mathcal{P} \cap R^{\times} \longleftarrow\!\!\shortmid R$$

Definition

If $T \subseteq \mathcal{P}$, the

$$\underline{\text{natural density of } T} := \lim_{x \to \infty} \frac{\#\{p \in T \quad : \quad p \leq x\}}{\#\{p \in \mathcal{P} \quad : \quad p \leq x\}}$$

Subring	S	H10	
$\mathbb{Q}$	$\mathcal{P}$	?	same answer
$\mathbb{Z}_{(p)}$	$\mathcal{P} \setminus \{p\}$	?	same answer
	co-finite	?	same answer
	density 1	**Theorem** NO sometimes	
$\mathbb{Z}[\frac{1}{p_1}, \ldots \frac{1}{p_n}]$	finite	NO	
$\mathbb{Z}$	$\emptyset$	NO	

Theorem (Poonen, 2002)

$\exists$ disjoint recursive subsets $T_1, T_2 \subseteq \mathcal{P}$ of natural density 0 such that if $T_1 \subseteq S \subseteq \mathcal{P} \setminus T_2$ and $R = \mathbb{Z}[S^{-1}]$ then

a) $\exists$ affine curve E'/R such that $\overline{E'(R)} \subseteq E'(\mathbb{R})$ has infinitely many connected components.

b) $\exists$ diophantine model of $\mathbb{Z}$ over R

c) H10/R has a negative answer

c.f. Schlapentokh [MRN, 2003]

Sketch of Proof

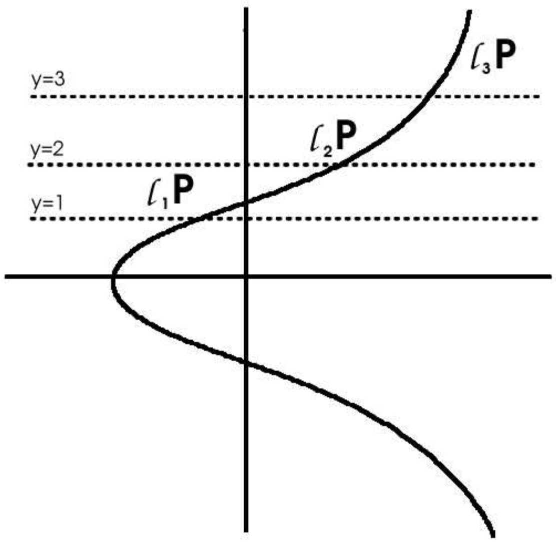
- Let E be the elliptic curve $y^2 = x^3 + x + 1/\mathbb{Q}$
- Key property: $E(\mathbb{Q})$ is of rank 1

- In fact, $E(\mathbb{Q}) \cong \mathbb{Z}$
- Let P be a generator of $E(\mathbb{Q})$
- Also $E(\mathbb{R}) \cong \mathbb{R}/\mathbb{Z}$
- $E' = E \setminus \{\text{point at } \infty\}$

Step 1 $\{lP \ : \ l \text{ is prime }\}$ is dense in $E(\mathbb{R})$

Proof:

- $E(\mathbb{R}) \xrightarrow{\sim} \mathbb{R}/\mathbb{Z}$
- P infinite order $\longleftrightarrow$ α irrational
- Use **Theorem(Vinogradov)** if $\alpha \in \mathbb{R}$ is irrational, then $\{l\alpha \ : \ l \text{ is prime }\}$ is dense in $\mathbb{R}/\mathbb{Z}$



Step 2

- Pick suitable primes $l_1 < l_2 < \ldots$ such that
$$|y(l_j P) - j| \le \frac{1}{10k}$$
$\forall j \ge 1$
- $y_j := y(l_j P)$

Step 3

- Choose T_1 so that $\pm l_j P \in E'(R)$
- Choose T_2 so that almost no other (finitely many) multiples of P are in $E'(R)$
- T_1 T_2 disjoint recursive and natural density 0

Step 4 Show that $A = \{y_1, y_2, \ldots\}$ is a diophantine model of $\mathbb{Z}_{>0}$ over R with:

$$\begin{aligned} \phi : \mathbb{Z}_{>0} &\longrightarrow A \\ j &\longmapsto y_j \end{aligned}$$

Proof

- $$A \doteq \underbrace{y(E'(R))}_{\text{diophantine}/R} \cap \mathbb{Q}_{>0}$$

  up to a finite set

- Therefore A is diophantine over R

- **Claim**

  The graph of $+$ on $\mathbb{Z}$ corresponds to a diophantine/R subset of A^3

- **Proof**

  $m+n=q \iff |y_m+y_n-y_q| \leq \frac{3}{10}$

- **<u>Claim</u>** The graph of $m \longmapsto m^2$ corresponds to a diophantine subset of A^2
- **<u>Proof</u>**

$$m^2 = q \iff |y_m^2 - \quad y_q| \le \frac{4}{10}$$

$$(m \pm \frac{1}{10m})^2$$

- $mn = q \iff (m+n)^2 = m^2 + n^2 + q + q$

- Diophantine model of $\mathbb{Z}_{>0}/R$
- H10/$\mathbb{Z}$ has a negative answer
- $\implies$ H10/R has a negative answer

- $K/\mathbb{Q}$
- $\text{density}(S) = 1 - \frac{1}{[K:\mathbb{Q}]}$
- $\mathcal{O}_K[S^{-1}] \cap \mathbb{Q} = \mathbb{Z}$