H10 Continued

Proposition

If $\mathbb{Z}$ is diophantine over $\mathbb{Q}$ then H10/$\mathbb{Q}$ has a negative answer.

Proof $\exists\ g$, say for $a \in \mathbb{Q}$, such that $a \in \mathbb{Z} \iff \exists\ \vec{x} \in \mathbb{Q}^n\ g(a, \vec{x}) = 0$.

Then H10/$\mathbb{Z}$ can be encoded as a subproblem of H10/$\mathbb{Q}$. Namely:

$\exists\ \vec{x} \in \mathbb{Z}^n\ f(\vec{x}) = 0 \iff$

$\exists \vec{x}, \vec{y_1}, \ldots, \vec{y_n} \in \mathbb{Q}$ such that:

$$\begin{cases} f(\vec{x}) = 0 \\ g(x_1, \vec{y_1}) = 0 \\ \vdots \\ g(x_n, \vec{y_n}) = 0 \end{cases}$$

Definition A diophantine model of the ring $\mathbb{Z}$ over $\mathbb{Q}$ is a diophantine set $S \subseteq X(\mathbb{Q})$ for some algebraic set X equipped with a bijection

$$\mathbb{Z} \xrightarrow[\sim]{\phi} S$$

such that the graphs of addition and multiplication on $\mathbb{Z}$ (subsets of $\mathbb{Z}^3$) correspond to diophantine sets in S^3

Remark ϕ is automatically recursive

- Search for the unique element of G_+ (the subset of S^3 corresponding to the graph of addition) of the form (a_0, a_0, a_0)
- Then $\phi(0) = a_0$

Proposition If $\exists$ a diophantine model of $\mathbb{Z}$ over $\mathbb{Q}$, then H10/$\mathbb{Q}$ has a negative answer.

Example

- $\exists\ E/\mathbb{Q}$ an elliptic curve with $E(\mathbb{Q}) \overset{\phi}{\underset{\sim}{\longleftarrow}} \mathbb{Z}$
- Is $(E(\mathbb{Q}), \phi)$ a diophantine model of $\mathbb{Z}/\mathbb{Q}$?

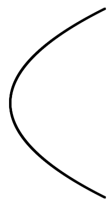
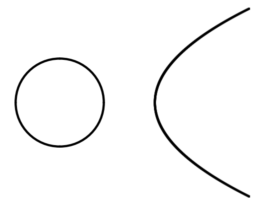
Mazur's Conjecture(MC)

If X is a variety over $\mathbb{Q}$ $\implies$ the topological closure of $X(\mathbb{Q})$ in $X(\mathbb{R})$ has at most finitely many connected components.

Theorem MC is true for curves/$\mathbb{Q}$.

Proof

- WLOG X is non-singular and projective
- WLOG $X(\mathbb{Q}) \neq \emptyset$
- **Genus 0:** Riemann-Roch $\implies X \cong \mathbb{P}^1$ and $\overline{\mathbb{P}^1(\mathbb{Q})} = \mathbb{P}^1(\mathbb{R})$

- **<u>Genus 1:</u>**
  - X is an elliptic curve
  - $X(\mathbb{Q})$ is a subgroup of $X(\mathbb{R})$
  - $\mathbb{R}/\mathbb{Z}$  or $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ 
  - $\overline{X(\mathbb{Q})}$ is a closed subgroup
  - closed subgroups are either finite or $\mathbb{R}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

- <u>**Genus $\geq$ 2:**</u>
  - $X(\mathbb{Q})$ is finite (Faltings)
  - $\overline{X(\mathbb{Q})} = X(\mathbb{Q})$

Proposition

- Assume MC
- X is a an algebraic set.
- $S \subseteq X(\mathbb{Q})$ is diophantine
- Then $\overline{S} \subseteq X(\mathbb{R})$ has finitely many connected components

Proof

- MC $\implies$ MC for algebraic sets
- Suppose $S \subseteq X(\mathbb{Q})$ is diophantine
- $\exists\, Y$ such that

$$\begin{array}{ccc} Y(\mathbb{Q}) & \subseteq & Y(\mathbb{R}) \\ \Big\downarrow\!\!\!\!\downarrow & & \Big\downarrow f \text{ continuous} \\ S & \subseteq & X(\mathbb{R}) \end{array}$$

- MC $\implies$ $\overline{Y(\mathbb{Q})}$ has finitely many components, say n of them
- $f\left(\overline{Y(\mathbb{Q})}\right)$ has $\leq n$ components
- $\overline{f\left(\overline{Y(\mathbb{Q})}\right)}$ has $\leq n$ components
- $\overline{f\left(\overline{Y(\mathbb{Q})}\right)} = \overline{S}$

$\mathbb{Z}$ diophantine/$\mathbb{Q}$ $\Longrightarrow$ $\exists$ diophantine model of $\mathbb{Z}/\mathbb{Q}$

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

MC false $\Longleftarrow$ H10/$\mathbb{Q}$ has a negative answer

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Proof of $\Leftarrow$

- Suppose $\exists$ diophantine model of $\mathbb{Z}/\mathbb{Q}$
- Then $\exists$ model of $\mathbb{N}$ over $\mathbb{Q}$, say $S \subseteq \mathbb{A}^n(\mathbb{Q}) = \mathbb{Q}^n$ with $\phi : \mathbb{N} \longrightarrow S$

- **Case 1** S is discrete in $\mathbb{R}^n$
  - each point of S is a connected component of $\overline{S}$
  - MC is false for S
  - MC is false

- **Case 2** S is non-discrete
  - $\exists$ non-isolated point $s \in S$
  - let $| \qquad |$ be the Euclidean distance on $\mathbb{R}^n$
  - We'll construct a sequence $m_0, m_1, \cdots \in \mathbb{N}$ such that $\phi(m_0), \phi(m_1), \ldots$ converges to s.
  - Let $m_0 = 0$
  - For $i > 1$, define $m_i :=$ smallest integer $> m_{i-1}$ such that $|\phi(m_i) - s| < \frac{1}{i}$
  - (s non-isolated $\implies$ m_i exists)

- Let $M = \{m_0, m_1, \ldots\} \subseteq \mathbb{N}$
- Then M is listable hence diophantine!
- Therefore $\phi(M)$ is diophantine
- $\phi(M)$ is a convergent sequence in $\mathbb{R}^n$
- $\overline{\phi(M)}$ has infinitely many connected components
- Contradicts MC

Question

- Suppose there is a diophantine interpretation of $\mathbb{Z}$ over $\mathbb{Q}$ (i.e. a diophantine set modulo a diophantine equivalence relation) that looks like $\mathbb{Z}$
- Would this contradict MC?