Pink-Roessler

- (K, σ) existentially closed difference field (saturated)
- Category of algebraic σ-varieties:
- **Objects** (V, ϕ)

  V irreducible variety$/K$

  $\phi : V \longrightarrow \sigma(V)$ dominant, defined over K

- ($\sigma-$) **Morphisms** $(V, \phi) \longrightarrow (W, \psi)$

$$\begin{array}{ccc} V & \xrightarrow[f]{} & W \\ \phi \big\downarrow & & \big\downarrow \psi \\ \sigma(V) & \xrightarrow{\sigma(f)} & \sigma(W) \end{array}$$

- σ-rational map
- algebraic σ-group: $\phi : G \longrightarrow \sigma(G)$ is an isogeny
- algebraic σ-subvariety

- $$\begin{aligned}(V,\phi)^{\#} &= \{a \in V(K) \;\mid\; \phi(a) = \sigma(a)\} \overset{\text{Zariski Dense}}{\subseteq} V(K) \\ &= \text{“finite dimensional” quantifier free definable} \\ &\ \text{set in } (K,\sigma)\end{aligned}$$

- Given (V,ϕ), (W,ψ) σ-varieties and $f : V \longrightarrow W$ a morphism of algebraic varieties.

  Then f is a σ-morphism $\iff f(V,\phi)^{\#} \subseteq (W,\psi)^{\#}$

- (V,ϕ) is trivial if V is defined over $k = \mathbf{Fix}(\sigma)$ and $\phi = \mathbf{id}$

Lemma Let (G, ϕ) be an algebraic σ-group with ϕ seperable.

Let $X \subseteq G$ be a σ-subvariety $(\phi|_X : X \longrightarrow \sigma(X))$ such that

$$\begin{aligned} \mathbf{Stab}_G(X) &= \{1\} \\ &= \{g \in G \quad | \quad gX = X\} \end{aligned}$$

Then $(X, \phi|_X)$ is σ-birational to a trivial algebraic σ-variety $(Y, \mathbf{id})$

Remark $(Y, \mathbf{id})^\# = Y(k)$

Comment This is closely related to a result of Ueno.

A complex torus.

$X \subseteq A$ analytic subvariety.

$\mathbf{Stab}(X) = \{1\} \implies X$ is algebraic.

Proof Writing G commutatively, WLOG assume $0 \in X$ by translation.

We have a version of a Gauss map $f : X \longrightarrow \mathbf{Gr}(V)$

- $\mathbf{j}_p(-)_a = \left(m_a/m_a^{p+1}\right)^*$
- For p large enough $\mathbf{j}_p(X-x)_o \subseteq V = \mathbf{j}_p(G)_o$ defines X
- $V = \mathbf{j}_p(G)_o$ is a K vector space
- $f(x) = \mathbf{j}_p(X-x)_o \subseteq V$

for large p $\quad f : X \xrightarrow[\sim]{\text{birational}} Y \subseteq \mathbf{Gr}(V)$

- ϕ seperable, $0 \in (G,\phi)^{\#} \implies$ we get an induced linear isomorphism:

$$\phi' : V \longrightarrow \sigma(V) = \mathbf{j}_p(\sigma(G))_o$$

- (V,ϕ') is a linear σ-variety
- $$\begin{aligned} (V,\phi')^{\#} &= \{v \in V \quad | \quad \phi'(v) = \sigma(v)\} \\ &= \text{finite dimensional vector space over } k = \mathbf{Fix}(\sigma) \end{aligned}$$

- Can choose a basis for V over K which is simultaneously a basis for $(V, \phi')^{\#}$ over k
- Thus assume $V = K^n$ and $(V, \phi')^{\#} = k^n$
- For $x \in (X, \phi)^{\#}$ then $(f(x), \phi')^{\#} \subseteq k^n$
- $\implies$ $f(x)$ is defined over k (i.e. $f(x) \in \mathbf{Gr}(V)(k)$) and Y is defined over k

- So $f\left((X,\phi)^{\#}\right) \subseteq Y(k)$ when Y is defined over k
- So f is a σ-birational isomorphism of (X,ϕ) and $(Y,1)$

Corollary (4.4) Let A be a semi-abelian variety, i.e. a commutative algebraic group which given by:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 1 \\ & & \| & & & & \| & & \\ & & \text{algebraic torus} \cong \mathbb{G}_m^k & & & & \text{abelian variety} & & \end{array}$$

Let $\phi : A \longrightarrow A$ be a seperable isogeny. Let $X \subseteq A$ be a subvariety with $0 \in X$, $\mathbf{Stab}(X)$ is trival, suppose that X generates A and X is ϕ-invariant. Then $\phi^n = \mathbf{id}$ for some n.

Proof

- Assume everything is defined over $\mathbf{Fix}(\sigma) = k$ where (K, σ) is an existentially closed difference field
- $A = \sigma(A)$ so (A, ϕ) is an algebraic σ-group, X is a σ-variety
- $(X, \phi) \xrightarrow[\sim]{\text{birational}} (Y, \mathbf{id})$
- **Fact (2.1)** $\implies$ $(A, \phi) \xrightarrow[\sim]{\sigma-\text{birational}} (B, \mathbf{id})$

  where B is a semi-algebraic group over k
- Uses

$$X^d \twoheadrightarrow A$$

$$Y^d \twoheadrightarrow$$

to define an equivalence relation

- $h : A/k \xrightarrow{\sim} B/k$ a σ-isomorphism
- So h is defined over $k_1 > k$, a finite extension
- $h(A, \phi)^{\#} \cong B(k)$
- $h^{-1} : B(k) \longrightarrow (A, \phi)^{\#} \subseteq A(k_1)$

- So for some m, $\sigma^m = \mathbf{1}$ on $(A, \phi)^{\#}$
- So $\phi^n = \mathbf{1}$ on $(A, \phi)^{\#} \implies \phi^n = \mathbf{1}_A$ because it is Zariski dense

Manin-Mumford Conjecture Characteristic 0, A semi-abelian variety, $X \subseteq A$. Then the Zariski closure $\overline{X \cap \mathbf{Tor}(A)}$ is a finite union of translates of semi-abelian subvarities of A

Step 1(Hiroshowski)

- Assume $0 \in X$, $X \cap \mathbf{Tor}(A)$ is Zariski dense in X.
- Assume everything is defined over a #-field.
- $\exists \sigma \in \mathbf{Gal}(\overline{\mathbb{Q}}/K)$ and monic polynomial $P(T) \in \mathbb{Z}[T]$ with no roots of unity among its roots and such that $\mathbf{Tor}(A) \subseteq \mathbf{Ker}\,(P(\sigma))$
- If $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$

$$\begin{aligned} P(\sigma) : A\left(\overline{\mathbb{Q}}\right) &\longrightarrow A\left(\overline{\mathbb{Q}}\right) \\ x &\longmapsto \sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_0 \end{aligned}$$