Model Theory and Diophantine Geometry Around Manin-Mumford

II "New" proof of M-M

III-V Drinfeld Modules version of M-M

I ACFA

- First Order Language L
- First Order Theory T in L
- M L-structure M model of T
- L =Language of Rings $+, -, \cdot, 0, 1$
- **ACF** =Theory of Algebraically closed fields
- $K \models$**ACF** K is a model of **ACF**

- K algebraically closed field
- V an irreducible variety defined over $k < K$
- $V = V(K)$
- A generic point of V over k is some $a \in V(K)$ such that $\mathbf{tr.deg} k(a)/k = \mathbf{dim} V$

- Fix L, T an $\forall\exists$ theory in L
- (F =Theory of fields)
- M is an existentially closed model of T, if $M \models T$ and whenever $M \subseteq N \models T$ then any quantifier formula $\Theta(x_1, \ldots, x_n)$ with parameters from M which is satisfied by some $(b_1, \ldots, b_n) \subseteq N$ is satisfied by some $(b'_1, \ldots, b'_n)$ from M
- $(N \models \exists x_1, \ldots, x_n \Theta \implies M \models \exists x_1, \ldots, x_n \Theta)$
- The extensionally closed (**ec**) models of F are the algebraically closed fields.

- L_σ language of $+$, $-$, $\cdot$, 0, 1, σ
- F_σ =Theory of algebraically closed fields with an automorphism in L_σ
- **Example**
    - $K(t)$ $\sigma(t) = t + 1$
    - $(\overline{\mathbb{Q}}, \sigma)$ σ is a Galois automorphism

- We say T has a model companion if the class of **ec** models of T is an elementary class, i.e. the class of models of a theory, say T'
- We say T' is the model companion of T

- Often T' has **QE**(quantifier elimination)
- For every L-formula $\Phi(x_1, \ldots, x_n)$ there is a **qf** (quantifier free) $\Phi'(x_1, \ldots, x_n)$ such that in any model of T

$$(\forall \vec{x})\,(\Phi(\vec{x}) \longleftrightarrow \Phi'(\vec{x}))$$

- (K, σ) $\quad\quad P\left(\vec{x}, \sigma(\vec{x}), \ldots, \sigma^n(\vec{x})\right) = 0$
- Solutions $\vec{x} \in K^m$ is a difference variety
- **Fact:** F_σ has a model companion which we call **ACFA** (algebraically closed field with an automorphism)
- **Axioms:**
  1. difference field
  2. algebraically closed
  3. Let V be irreducible variety over K. Let $W \subseteq V \times \sigma(V)$ be an irreducible sub-variety projecting dominantly on V and $\sigma(V)$, then $\exists (a, \sigma(a)) \in W(K)$

Special Case

$\phi : V \longrightarrow \sigma(V)$ is a dominant morphism. The Axioms imply that

$$(V,\phi)^{\#} \underset{\text{def}}{=} \{a \in V(K) \quad : \quad \phi(a) = \sigma(a)\}$$

is Zariski dense in V, when B is a finite dimensional difference algebraic variety.

$(V,\phi) \longrightarrow (W,\psi)$ $\qquad\qquad$ (K,σ) or $(\mathbb{C},1)$

Properties

- **ACFA** is not a complete theory
- $\mathbf{Th}(K, \sigma)$ determined by isomorphism type of $(\overline{K_o}, \sigma|_{\overline{K_o}})$ where K_o is the prime field
- **ACFA** does not have **QE**

  If (K, σ) is a model, then $F(\sigma)$ is a psuedo-finite field.

- But **ACFA** has near **QE**
- Let $\psi(x_1, \ldots, x_n)$ be a L_σ-formula. Then it is equivalent in models of **ACFA** to

$$\exists y \left(\Theta\left(\vec{x}, \sigma(\vec{x}), \ldots, \sigma^m(\vec{x}), y, \sigma(y), \ldots, \sigma^m(y)\right)\right)$$

where $\Theta(\vec{z}, \vec{w})$ is a finite conjunction of polynomial equations over $\mathbb{Z}$ and

$$\Theta\left(\vec{z}, \vec{w}\right) \implies \vec{w} \in \mathbf{acl}(\vec{z})$$

Crucial Objects are finite-dimensional definable set in $(K, \sigma) \models \mathbf{ACFA}$

Let $Z \subseteq K^n$ be definable over $k < K$ a difference subfield.

Z is finite dimension if $\exists$ finite bound on

$$\mathbf{tr.deg}\big(k(a, \sigma(a), \sigma^2(a), \ldots)/k\big)$$

for $a \in Z$ as a varies in Z