Presburger

- Considered: $\mathbb{Z}$, $\pm$, 0, 1, $<$
- Basic sets in $\mathbb{Z}^n$ either:

  Zero sets of linear functions$/_{\mathbb{Z}}$

  or Positivity sets of linear functions$/_{\mathbb{Z}}$

  or Congruence sets of linear functions$/_{\mathbb{Z}}$

  $f(\overline{x}) \equiv 0 \,\mathbf{mod}\, m$

- Take boolean combinations $\rightsquigarrow$ these ultimately will be definable sets (1929)
- Presburger's methods eventually showed the class is closed under projection
- First serious result in model theoretic algebra
- What about multiplication?
- Depends what you mean– The exact analog would use $\mathbb{Z}, \cdot, <$ and there is no such result

  1. $\cdot, <$ No such result (Structure is undecidable)
  2. $\cdot$ alone Yes, Skölem (Basic sets are not easy to describe)

Gödel (1931)

Basic Situation

zero sets of polynomials over $\mathbb{Z}$

↓

Boolean Combinations

↓ project

Boolean combinations

↓ project

⋮

This can go on forever in a “complicated” or Gödelian situation

Generating sets like this:

$$\underbrace{\forall \overline{x_1} \exists \overline{x_2} \forall \overline{x_3} \cdots}_{\text{length } k \text{ (prefix)}} P(\qquad\qquad\qquad) = 0$$

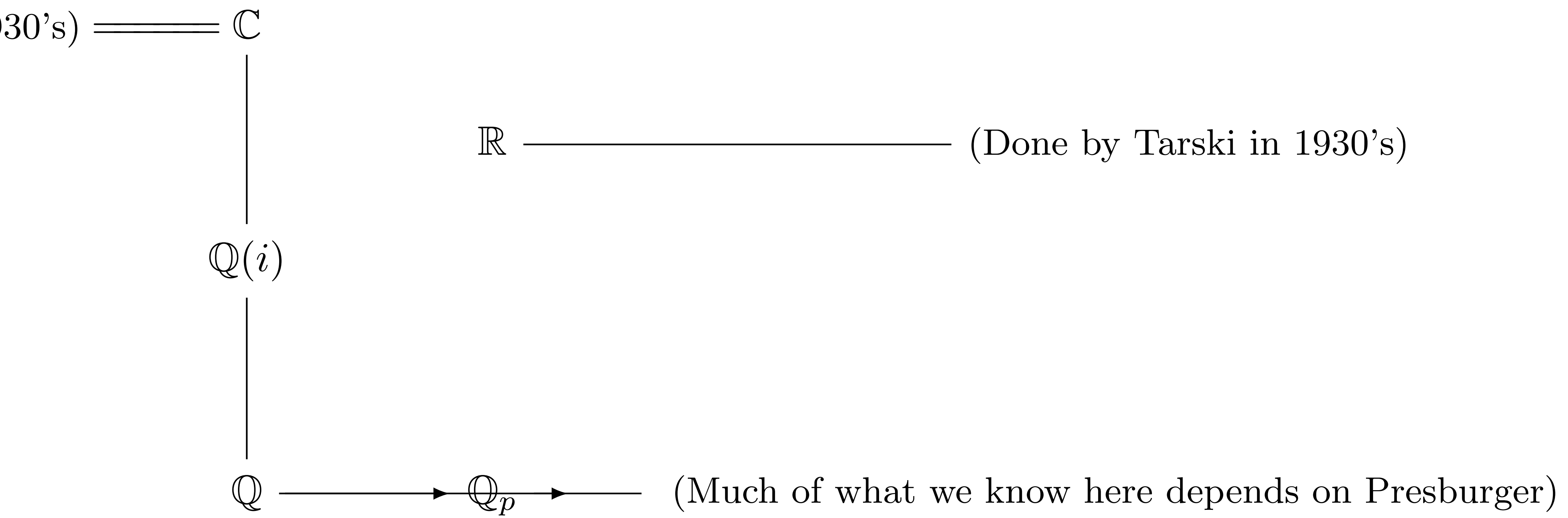
Should be a system of polynomials but $\exists$ tricks

Basic result Get new sets at each level except in Skökem's and Presburger's

Point is $(\mathbb{Z}, +)$ get basic sets, $\cdot$ don't!

Non-Gödelian Situations

Usually "completions"



$\mathbb{R}$:

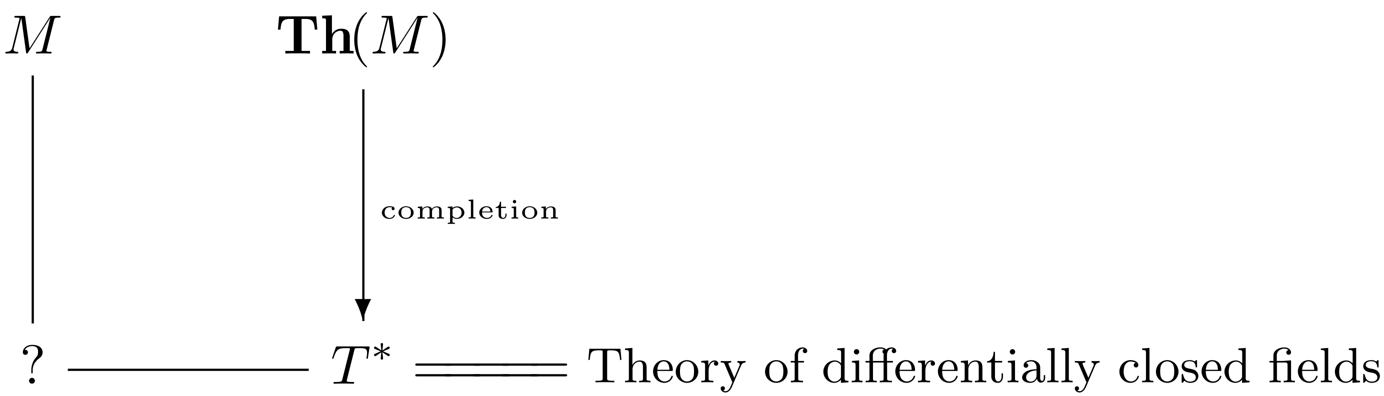
Basic sets:

$$\left\{\begin{array}{l}\text{Zero sets}\\ \text{Positivity Sets}\end{array}\right. \quad \text{for } f \in \mathbb{R}[\overline{x}]$$

Boolean Operations: Semi-algebraic sets

"Closed under projection"

Ring of algebraic integers


M
Th(M)
completion
?
T*
Theory of differentially closed fields