- k a **char** 0 field
- X a smooth variety$/_k$ of dimension d
- $f : X \longrightarrow \mathbb{A}^1_k$
- Look at $\mathcal{Y}_n := \{y \in \mathcal{L}(X) \mid \mathbf{val}(f(\phi)) = n\}$
- $\mathcal{Y}_n$ depends only on image of ϕ in $\mathcal{L}_n(X)$

- $Y_n := \pi_n^{-1}(\mathfrak{X}_n)$
- $\mathfrak{X}_n := \{Y \in \mathcal{L}_n(X) \mid \mathbf{val} f(\phi) = n\}$
- $\mu(Y_n) = \frac{[\mathfrak{X}_n]}{\mathbb{L}^{(n+1)d}}$
- $\mathcal{L}(X)(K) = X(K[\![t]\!])$
- $\mathcal{L}_n(X) = X\left(K[\![t]\!]/_{t^{n+1}K[\![t]\!]}\right)$
- $\mathcal{L}_0(X) = X \quad \mathcal{L}_1(X) = \mathbf{T}X$

- Consider $\mathcal{Z}_{\mathbf{mot}} = \sum_{n \geq 1} \mathbb{L}^{-nd} T^n$
  motivic analog of Igusa's series
- Consider $h : Y \longrightarrow X$ **log**-resolution of $f = 0$
- $\left| h^{-1}\left(f^{-1}(0)\right) \right| = \bigcap_{i \in J} E_i$
- E_I, E_I^o, $I \subseteq J$
- $h^{-1}\left(f^{-1}(0)\right) = \sum_{i \in J} N_i E_i \qquad \nu_i$

Proposition (Denef-Loeser)

$$\mathcal{Z}_{\mathbf{mot}} = \mathbb{L}^{-d} \sum_{I \subseteq J} [E_I^o] \prod_{i \in I} \frac{(\mathbb{L}-1)\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}$$

Proof Follows from the change of variables formula

- Setting $T = \mathbb{L}^{-s}$ for $s \in \mathbb{N}$

$$\mathcal{Z}_{\mathbf{mot}}\left(\mathbb{L}^{-s}\right) = \mathbb{L}^{-d} \sum_{I \subseteq J} [E_I^o] \frac{(\mathbb{L}-1)\mathbb{L}^{-\nu_i - sN_i}}{1 - \mathbb{L}^{-\nu_i - sN_i}}$$

- $\implies \mathcal{Z}_{\mathbf{mot}}\left(\mathbb{L}^{-s}\right) \in \mathcal{M}_k[\frac{1}{[\mathbb{P}_k^i]}]_{i \geq 1} =: \mathcal{M}_{l,\mathbf{loc}}$
  where $[\mathbb{P}_k^i] = 1 + \mathbb{L} + \cdots + \mathbb{L}^i$
- **Eu** may be extended to $\mathcal{M}_{k,\mathbf{loc}}$ $\quad \mathbf{Eu} : \mathcal{M}_{k,\mathbf{loc}} \longrightarrow \mathbb{Q}$
- $\implies \mathbf{Eu}(\mathcal{Z}_{\mathbf{mot}}\left(\mathbb{L}^{-s}\right)) = \mathcal{Z}_{\mathbf{top},f}$

Remark 1

- $$\mathbf{Eu}(\mathfrak{X}_n) = \bigwedge (\mathbf{M}^n, \mathbf{H}^\bullet(\mathbf{F}, \mathbb{C}))$$
- $\mathbf{M}^n$ is the monodromy operator
- $\mathbf{F}$ is the Milnor fibre at 0
- $$\wedge(\mathbf{M}^n) = \sum (-1)^{-i} \mathbf{Tr}\big(\mathbf{M}^n, \mathbf{H}^i\big)$$

Remark 2

$$\mathfrak{X} \xrightarrow{\quad f_n \quad} \mathbb{G}_{m,k} = \mathbb{A}^1_k \setminus 0$$

$$\phi \longmapsto \mathbf{ac}(f(\phi)) = \text{coefficient of } t^n \text{ in } f(\phi)$$

Contains a lot of information

X, X' birational Calabi-Yau

$$\begin{array}{ccc} & X'' & \xrightarrow{\quad h' \quad} \\ X & \xleftarrow{\quad h \quad} & & X' \end{array}$$

X'' smooth

h, h' birational

Compute

$$\begin{array}{ccc} \mu\left(\mathcal{L}(X)\right) & = & \displaystyle\int_{\mathcal{K}(X'')} \mathbb{L}^{-\mathbf{Ord}\left(\mathbf{jac}(h)\right)} \\ \Big\| & & \Big\| \text{ from hypothesis} \\ \mu\left(\mathcal{L}(X')\right) & = & \displaystyle\int_{\mathcal{K}(X'')} \mathbb{L}^{-\mathbf{Ord}\left(\mathbf{jac}(h')\right)} \end{array}$$

- $\implies \mu\left(\mathcal{L}(X)\right) = \mu\left(\mathcal{L}(X')\right)$
- $\implies \frac{[X]}{\mathbb{L}^d} = \frac{[X']}{\mathbb{L}^d}$
- X smooth so $\mu\left(\mathcal{L}(X)\right) = \frac{[X]}{\mathbb{L}^d}$
- So $[X] = [X']$ in $\overline{\mathcal{M}}_k$

If X , X' are k-equivalent, meaning that:

$$\exists \qquad X'' \xrightarrow{\;h'\;}$$

$$X \xleftarrow{\;h\;} \qquad X'$$

such that $h^*\Omega^d_X = {h'}^*\Omega^d_{X'}$

Then $\implies$ $[X] = [X']$ in $\overline{\mathcal{M}}_k$

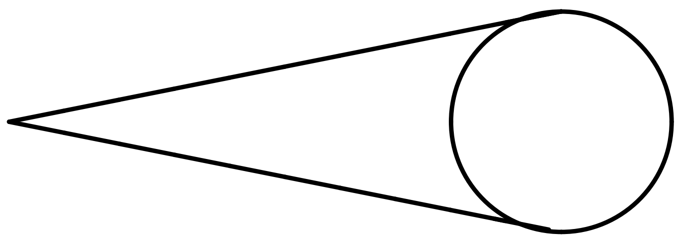
- Take X smooth, $h : Y \longrightarrow X$ with exceptional locus DNC
- $\mu\left(\mathcal{L}(X)\right) = \mathbb{L}^{-d} \sum_{I \subseteq J} [E_I^o] \prod \frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i}-1} = \frac{[X]}{\mathbb{L}^d}$
- $\implies \mathbf{Eu}(X) = \sum_{I \subseteq J} \mathbf{Eu}(E_I^o) \prod_{i \in I} \frac{1}{\nu_i}$
- First proved by p-adic integration
- Also follows from weak factorization

If X is singular, similarly

- $\mu(\mathcal{L}(X)) = \mathbb{L}^{-d} \sum E_I^o \prod \frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i}-1}$
- $\mathbf{Eu}(\mu(\mathcal{L}(X))) \in \mathbb{Q}$

- $h : Y \longrightarrow X$ birational proper
- $Y \setminus E \cong X \setminus F$ $\qquad$ E,F codimension 1
- $\mathcal{L}(Y) \setminus \mathcal{L}(E) \cong \mathcal{L}(X) \setminus \mathcal{L}(F)$
- $\mathcal{L}(E)$, $\mathcal{L}(F)$ have infinite co-dimension

In $\mu(\mathcal{L}(X))$ $X_{\mathbf{sing}}$ counts:



$\mathcal{L}(X_{\mathbf{sing}})$ has infinite co-dimension but arcs with origin in our $X_{\mathbf{sing}}$ and generic point outside count

For free:

- If $X = \mathbb{A}_k^d$ and $K = \#$-field
- For all most all $\mathfrak{p}$

$$N_{\mathfrak{p}}(\mathcal{Z}_{\mathbf{mot},f})(T) = \int_{R_{\mathfrak{p}}^d} |f|_{\mathfrak{p}}^s |dx|_{\mathfrak{p}}$$

- $T = q^{-s}$ $\qquad q = |k_{\mathfrak{p}}|$

Also you have $Q_{\mathbf{mot}}$ a rational series in $\mathcal{M}_K[\![T]\!]$ such that

$$N_{\mathfrak{p}}(Q_{\mathbf{mot}}) = Q_{\mathfrak{p}}$$

for almost all $\mathfrak{p}$ where $Q_{\mathfrak{p}}$ is the corresponding Igusa series on $K_{\mathfrak{p}}$

- **Question:** Do we have a similar result for Serre's series?
- Take X a variety over $\mathbb{Z}_p$
- $\mathcal{N}_n := \left| X\left(\mathbb{Z}_p/_{p^{n+1}\mathbb{Z}_p}\right)\right|$
- $\tilde{\mathcal{N}}_n := \left|\mathbf{Im}\,(X(\mathbb{Z}_p)) \text{ in } X\left(\mathbb{Z}_p/_{p^{n+1}\mathbb{Z}_p}\right)\right|$
- $Q(T) = \sum_{n\geq 0} \mathcal{N}_n T^n$
- $P(T) = \sum_{n\geq 0} \tilde{\mathcal{N}}_n T^n$

Geometric Analogue of $\tilde{\mathfrak{N}}_n$

$$\begin{array}{ccc} X(\mathbb{Z}_p) & \longrightarrow & X\left(\mathbb{Z}_p/_{p^{n+1}\mathbb{Z}_p}\right) \\ \updownarrow & & \\ \mathcal{L}(X) & \xrightarrow{\pi_n} & \mathcal{L}_n(X) \end{array}$$

Consider $P_{\mathbf{mot}}(T) = \sum_{n\geq 0} \left[\pi_n\left(\mathcal{L}(X)\right)\right] T^n$

Remark: $[\pi_n(\mathcal{L}(X))]$ makes sense since $\pi_n(\mathcal{L}(X))$ is constructible

3 Proofs

1. Pas Quanitfier Elimination
2. Hironaka
3. Greenberg's Theorem

Theorem (Greenberg) $\forall n \geq 0 \; \exists \gamma(n) \geq n$ such that

$$\pi_n \left(\mathcal{L}(X) \right) = \mathbf{Im} \left(\pi_{\gamma(n)} \mathcal{L}_{\gamma(n)}(X) \longrightarrow \pi_n \mathcal{L}_n(X) \right)$$

Theorem (Denef-Loeser) The series $P_{\mathbf{mot}}(T)$ is rational in $\mathcal{M}_k[\![T]\!]$

- Take X defined over K a number field
- In general $N_{\mathfrak{p}}(P_{\mathbf{mot}}) \neq P_{\mathfrak{p}}(T)$
- $\phi \in \mathfrak{L}_n(X)$
- $\phi \in \mathfrak{L}_n(X)(K)$
- $\phi \in \pi_n\left(\mathfrak{L}(X)\right)$ if $\exists \psi \in \mathfrak{L}(X)(K')$ such that $\pi_n(\psi) = \phi$