- k a field
- $\mathbf{Var}_k$ =varieties/k =schemes of finite type/k separated and reduced
- $\mathbf{K}_o(\mathbf{Var}_k)$ quotient of the free abelian group on isomorphism classes of $\mathbf{Var}_k$ by the relation $[S] = [S'] + [S \setminus S']$ for S' closed in S
- Ring structure on $\mathbf{K}_o(\mathbf{Var}_k)$ $[S][S'] = [S \times S']$

- $\mathbf{K}_o(\mathbf{Var}_k)$ is universal additive (and multiplicative) invariant
- **Example:** $k = \mathbb{C}$ $\quad \mathbf{Eu} : \mathbf{K}_o(\mathbf{Var}_k) \longrightarrow \mathbb{Z}$

Example: Hodge polynomial for **Char** $k = 0$

- X is smooth and projective
- $h^{p,q} := \mathbf{dim}_k \mathbf{H}^q(X, \Omega^p)$
- $H(X) := \sum (-1)^{p+q} h^{p,q}(X) u^p v^q \quad \in \mathbb{Z}[u,v]$
- Deligne $\implies$ H has unique extension to a ring morphism $H : \mathbf{K}_o(\mathbf{Var}_k) \longrightarrow \mathbb{Z}[u,v]$
- $k = \mathbb{C} \quad \mathbf{Eu}(X) = H(X)(1,1)$
- Bittner (using weak factorization) gives very efficient criteria to extend an invariant from smooth projective schemes to $\mathbf{K}_o(\mathbf{Var}_k)$ (**Char** $k = 0$)
- $E \subseteq X$, $\tilde{E} \subseteq \mathbf{Bl}_E X = Y \quad \mathcal{X}(X) - \mathcal{X}(E) = \mathcal{X}(Y) - \mathcal{X}(\tilde{E})$

- **Example:** Invariant for **Char** $k = 0$

$$\mathcal{X}_C : \mathbf{K}_o(\mathbf{Var}_k) \longrightarrow \mathbf{K}_o(\mathbf{ChMot}_k)$$

(Gillet/Soulé, Guillen/Navaro-Aznev)

- **Example:** $k = \mathbb{F}_q$

$$\mathbf{N} : \mathbf{K}_o(\mathbf{Var}_k) \longrightarrow \mathbb{Z}$$

$$[S] \longmapsto |S(\mathbb{F}_q)|$$

- **Example:** $k = \#$-field, S variety/k

$\mathbf{N}_\mathfrak{p}(S) := |S(k_\mathfrak{p})|$ defined for almost all $\mathfrak{p}$

$\mathbf{N} : \mathbf{K}_o(\mathbf{Var}_k) \longrightarrow \frac{\prod_\mathfrak{p} \mathbb{Z}}{\oplus \mathfrak{p}\mathbb{Z}}$

Theorem (Poonen):

If **Char** $k = 0$ then $\mathbf{K}_o(\mathbf{Var}_k)$ is not a domain

Uses factorization by $\mathbb{L}$ $\quad \mathbf{K}_o(\mathbf{Var}_k) \longrightarrow \mathbf{K}_o(\mathbf{Var}_k)/_{\mathbb{L}}$

$$\mathbf{K}_o(\mathbf{Var}_k) \longleftrightarrow \mathbb{Z}$$

$$\mathcal{M}_k \longleftrightarrow \mathbb{Z}[\frac{1}{p}]$$

$\mathcal{M}_k = \mathbf{K}_o(\mathbf{Var}_k)[\mathbb{L}^{-1}] \qquad \mathbb{L} := [\mathbb{A}^1_k]$

- $\Theta : \mathbf{K}_o(\mathbf{Var}_k) \longrightarrow \mathcal{M}_k$
- Don't know whether Θ is injective or not

- take X to be a variety over k
- $X^{(n)} = \overbrace{X \times \cdots \times X}^{n-\text{times}} /_{\sigma_n}$
- Kapranov: Is $Z(T) := \sum_{n \geq 0} [X^{(n)}] T^n$ a rational series?
- Larsen-Lunts: For most surfaces ($\mathbf{Char}\ \ k = 0$) $Z(T)$ is rational in $\mathbf{K}_o(\mathbf{Var}_k)[\![T]\!]$
- Uses $\mathbf{K}_o(\mathbf{Var}_k)/\mathbb{L}$
- Question: Is $Z(T)$ rational in $\mathcal{M}_k[\![T]\!]$

Lemma

If $k = \mathbb{F}_q$, then $\mathcal{N}(Z(X)(T)) = Z_{HW}(X)(T) \in \mathbb{Z}[\![T]\!]$ where:

$$Z_{HW}(X)(T) = \mathbf{exp}\left(\sum_{n \geq 1} \frac{N_n(X)}{n} T^n\right)$$

with $N_n = |X(\mathbb{F}_{q^n})|$

(Dwork) Z_{HW} is rational

$$\mathbb{Q}_p \longleftrightarrow k((t)) \qquad \mathbb{C}((t))$$

$$\mathbb{Z}_p \longleftrightarrow k[\![t]\!]$$

Integration on $\mathbb{C}((t))^m$ with values in $\mathcal{M}_k$ or $\hat{\mathcal{M}}_k$ (Kontsevich)

$$\begin{array}{ccc} \mathbf{K}_o(\mathbf{Var}_k) & \longleftrightarrow & \mathbb{Z} \\ \mathcal{M}_k & \longleftrightarrow & \mathbb{Z}[\frac{1}{p}] \\ \big\downarrow & & \big\downarrow \\ \hat{\mathcal{M}}_k & \longleftrightarrow & \mathbb{R} \end{array}$$

- In $\mathbb{R}$ $\quad p^{-i} \longrightarrow 0$ as $i \longrightarrow \infty$
- In $\hat{\mathcal{M}}_k$ $\quad \mathbb{L}^{-i} \longrightarrow 0$ as $i \longrightarrow \infty$

- $F^m\mathcal{M}_k$ subgroup generated by $\frac{[S]}{\mathbb{L}^i}$ with $i - \mathbf{dim} S \leq -m$
- $\frac{1}{\mathbb{L}^m} \in F^{-m}\mathcal{M}_k$ and $F^{m+1} \subseteq F^m$
- $F^m F^{m'} \subseteq F^{m+m'}$
- $\hat{\mathcal{M}}_k$ is the completed ring with respect to the filtration F
- Not known whether $\Theta : \mathcal{M}_k \longrightarrow \hat{\mathcal{M}}_k$ is injective or not
- $\overline{\mathcal{M}}_k := \Theta(\mathcal{M}_k)$
- H (so **Eu**) factors through $\overline{\mathcal{M}}_k$

- X variety over k
- $X\left(k[\![t]\!]\right)$
- $\mathcal{L}\left(X\right)$ k-scheme such that for every field $K \supseteq k$

  $\mathcal{L}(X)(K) = X(K[\![t]\!])$
- If $X = \mathbb{A}_k^m = \mathbf{Spec}(k[x_1, \ldots, x_m]) \quad x_i = \sum_{j=0}^{\infty} a_{ij} t^j$
- $\mathcal{L}(X) = \mathbf{Spec}(k[a_{ij}])$
- If $X \subseteq \mathbb{A}_k^m$ is defined by $f_\alpha = 0$ then $\mathcal{L}(X)$ is defined by

  $f_\alpha\left(x_1(t), \ldots, x_m(t)\right) = 0$
- $\implies$ equations in terms of the a_{ij}'s

- Replacing $k[\![t]\!]$ by $k[\![t]\!]/_{t^n}$ we get $\mathcal{L}_n(X)$ which is of finite type over k
- $\pi_n : \mathcal{L}(X) \longrightarrow \mathcal{L}_n(X)$

- Assume X is smooth
- Take $A \subseteq \mathcal{L}_n(X)$ a constructible set
- **Remark**: $[A] \in \mathbf{K}_o(\mathbf{Var}_k)$
- $C := \pi_n^{-1}(A)$ cylinder
- $\mu(C) := \frac{[A]}{\mathbb{L}^{(n+1)d}}$ where $d = \mathbf{dim} X$

- If A' =pre-image of A in $\mathcal{L}_{n+1}(X)$ $\quad C = \pi_{n+1}^{-1}(A')$
- Should check: $\frac{[A']}{\mathbb{L}^{(n+1)d}} = \frac{[A]}{\mathbb{L}^{(n+1)d}}$
- This is OK because X being smooth, $\mathcal{L}_{n+1}(X) \longrightarrow \mathcal{L}_n(X)$ is a Zariski function with fibre $\mathbb{A}_k^d$

- We can arrange a measure for semi-algebraic subsets of $\mathcal{L}(X)$
- Assume $X = \mathbb{A}_k^m$
- Consider Pas language:

Sorts	$\longrightarrow$	K	value field	ring language
	$\longrightarrow$	k	residue field	ring language
	$\longrightarrow$	Γ	value group	Presburger language

- $+ \;\; \cdot \;\; \mathbf{val} : k \setminus 0 \longrightarrow \Gamma$
-

$$\cdot\mathbf{ac} : \quad K \longrightarrow k$$

$$x \longmapsto \text{residue class of } t^{-\mathbf{val}(x)} x \text{ in } k$$

$$0 \text{ if } x = 0$$

- If **Char** $k = 0 \quad K = k((t))$ satisfies QE in Pas language
- Semi-algebraic subsets of X are the definable subsets in Pas language
- Semi-algebraic subsets have a measure in $\hat{\mathcal{M}}_k$

Theorem (Denef-Loeser)

Consider $h : Y \longrightarrow X$ a birational proper morphism of varieties with Y smooth. Take $A \subseteq \mathcal{L}(X)$ measurable, $\alpha : A \longrightarrow \mathbb{N}$ a function then

$$\int_A \mathbb{L}^{-\alpha} d\mu = \int_{h^{-1}(A)} \mathbb{L}^{-\alpha \circ h - \mathbf{ord}(\mathbf{jac}(h))} d\mu$$

where

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{i \geq 0} \mathbb{L}^{--i} \mathbf{vol}\left(\alpha^{-1}(i)\right) \quad \in \hat{\mathcal{M}}_k$$

and $\mathbf{ord}(\mathbf{jac}(h))$ at an arc $\phi \in \mathcal{L}(Y)$ is the order of the jacobian of h at ϕ.

$h : \mathcal{L}_n(Y) \longrightarrow \mathcal{L}_n(X)$