Motivic and p-adic Integraion

- Andre Weil (1950's)
- Siegel-Weil Formulas
- Tamagawa Numbers

p-adic Integration

- $K = \mathbb{Q}_p \qquad R = \mathbb{Z}_p \qquad \mathfrak{p} = p\mathbb{Z}_p$
- K finite extension of $\mathbb{Q}_p$
- $\mathbf{val}: K \longrightarrow \mathbb{Z} \cup \{\infty\}$
- $|x| = q^{-\mathbf{val}(x)}$ where $q = |R/\mathfrak{p}|$
- K is $\mathbb{Z}$-values complete with finite residue field and is totally disconnected

- $(K^n, +)$ is a locally compact (abelian) group
- μ_n Haar measure on K^n, translation invariant and unique if you ask $\mu_n(R^n) = 1$
- ϖ a uniformizing parameter, $\mathbf{val}(\varpi) = 1$
- $\mu_n(a + \varpi^m R^n) = q^{-mn}$ Ball of radius q^{-m}

$f : K^n \longrightarrow K^n$ K-analytic function

$$\int_A |f|^s d\mu_n = \sum_{i \in \mathbb{Z}} \mu_n \left(\mathbf{val}(f) = i\right) q^{-is}$$

If the sum converges in $\mathbb{R}$

Integration on open $U \subseteq K^n$

If $\phi : U \xrightarrow{\sim} V$ is an analytic isomorphism between two open sets, we have the change of variable formula (CVF):

$$\mu_n|_U = \left|\mathbf{det}\left(\frac{\partial \phi_i}{\partial x_j}\right)\right|^{-1} \phi^* \left(\mu_n|_V\right)$$

- Can use the CVF to integrate smooth analytic varities
- X smooth n-dimensional analytic variety$/K$
- ω a gauge form on X – a top degree nowhere vanishing differential form
- $\int_X |\omega|$ Use local charts $X \supseteq U \xleftarrow{\backsim} V \subseteq K^n$
- $\phi^* \omega|_U = f\ dx_1 \wedge \cdots \wedge dx_n$

Take $f \in R[x_1, \dots, x_m]$

$N_n = \left|(x_1, \dots, x_m) \in R/\mathfrak{p}^{n+1}R \ \mid\ f(x_1, \dots, x_m) \equiv 0 \pmod{\mathfrak{p}^{n+1}}\right|$

Problem (Borevich-Shaferevich)

Is the series $Q(T) = \sum_{n \geq 0} N_n T^n \in \mathbb{Z}[[T]]$ rational?

Theorem (Igusa '76-'78)

Assume **Char** $K = 0$ then $Q(T)$ is rational.

Sketch of Proof

- Compare $Q(T)$ with $I(s) = \int_{R^m} |f|^s d\mu_n$.
- Since

  $$N_n = q^{(n+1)m} \mu_n \left(\{ x \in R^m \ \mid \ \mathbf{val}(f(x)) \geq n+1 \} \right)$$

  And

  $$Q\left(q^{-m-s}\right) = \frac{q^m}{q - q^{-s}} \left(1 - I(s)\right)$$

  Enough to prove $I(s)$ is rational function of q^{-s}
- Use Hironaka's resolution to get $h : X \longrightarrow R^m$ where X is smooth, h is birational, proper and a isomorphism outside of $h^{-1}\left(f^{-1}(0)\right)$

- $h^{-1}\left(f^{-1}(0)\right)$ is a divisor with normal crossings
- In particular, locally on X, $f \circ h = u \cdot y_1^{N_1} \cdots y_m^{N_m}$ with u a unit and $y_1, \ldots, y_m$ local coordinates.
- By CVF $I(s)$ may be expressed as a sum of integrals of the form:

$$\int_{\mathbf{val}(y_i) \geq s} \prod |y_i|^{N_i s + m_i} |dy_1 \ldots dy_m|$$

- Here the m_i's are defined by:

$$h^* \, dx_1 \wedge \cdots \wedge dx_n = v \prod y_i^{m_i} \, dy_1 \wedge \cdots \wedge dy_m$$

where v is a unit

Question **(Serre)** Take $f \in R[x_1, \ldots, x_m]$,

$$\tilde{N}_n = \left| \left\{ (x_1, \ldots, x_m) \in \big(R/\mathfrak{p}^{n+1}R\big) \;\middle|\; \begin{array}{l} \exists (y_1, \ldots, y_m) \in R^m \\ \text{such that } f(y_1, \ldots, y_m) = 0 \\ \text{and } x \equiv y \pmod{\mathfrak{p}^{n+1}} \end{array} \right\} \right|$$

Is $P(T) = \sum_{n \geq 0} \tilde{N}_n T^n$ rational?

Theorem (Denef, '82)

If **Char** $K = 0$, then $P(T)$ is rational

Proof

- Express $P(T)$ as an integral
- $X := \{f = 0\}$ in R^m
- $$J(s) = \int_{R^m} \mathbf{d}(x, X)^s d\mu_m$$

  where **d** is the distance function.
- Enough to prove $J(s)$ is a rational function of q^{-s}

- Denef uses the fact that the distance function is semi-algebraic.
- Definable function $\mathbb{Q}_p^m \longrightarrow \mathbb{Z}$ means that the graph is definable
- **Macintyre's Quantifier Elimination**
  - Assume $K = \mathbb{Q}_p$
  - $\mathcal{L}_{\mathbf{Mac}} = 1^{\text{st}}$-order languange $\quad +, -, \times, 0, 1$
  - For $d \geq 2$, P_d is the predicate: x is a d^{th} power of an element in $\mathbb{Q}_p$
  - Macintyre's Elimination: $\mathbb{Q}_p$ has quantifier elimination in $\mathcal{L}_{\mathbf{Mac}}$

- Uses $\mathfrak{L}_{\mathbf{Pres}}$ Presburger languange $+$, $\leq$, 0, 1, $(\quad \equiv \quad \pmod{d})$
- Consider the languange $\mathfrak{L}$:
  1. With two sorts of variables:
     - Variables running over $\mathbb{Q}_p$ $\qquad \mathfrak{L}_{\mathbf{Mac}}$
     - Variables running over $\mathbb{Z}$ $\qquad \mathfrak{L}_{\mathbf{Pres}}$
  2. $\mathbf{val} : K \setminus \{0\} \longrightarrow \mathbb{Z}$

- Can consider definable subsets of $\mathbb{Q}_p^m \times \mathbb{Z}^n$ in $\mathcal{L}$
- **Remark:** $\mathcal{L}$-definable sets of $\mathbb{Q}_p^m$ are exactly the semi-algebraic subsets of $\mathbb{Q}_p^m$
- **Remark:** If Hironaka's Theorem (in its strong form) is known in **Char** p then Q is rational in **Char** p by Igusa's proof
- **Remark:** The rationality of P is a much more open problem – no known conjectures imply it