# Arizona Winter School lectures

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# LECTURE 2

# Lecture 2

## 2.1 The families

In this lecture we study the families $f : X \to S$ of nonsingular projective curves with affine equation

$$X_t : \quad y^p = x(x-1)(x-t)$$

over the base $S = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ where $p$ is a prime $\geq 5$. By Riemann-Hurwitz

$$\text{genus } X_t = p-1,$$

Hence

$$V_{\mathbb{Z}} = R^1 f_* \mathbb{Z}$$

is a rank $2(p-1)$ weight 1 PVHS over $S$

Consider the automorphism over $S$

$$\begin{tikzcd} X \arrow[rr, "\sigma"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd} \qquad (x,y) \longmapsto (x, \exp(\tfrac{2\pi i}{p})y)$$

This induces an automorphism

$$\sigma = \sigma^* : \mathbb{V}_{\mathbb{Z}} \longrightarrow \mathbb{V}_{\mathbb{Z}}$$

of PVHS's.

<u>Proposition 2.1.1</u> For any $t \in S$.

(a) $\left(\mathbb{V}_{t,\mathbb{Z}}\right)^{\sigma=1} = (0)$

(b) The eigenvalues of $\sigma$ on

$$V_t^{1,0} = H^0(X_t, \Omega^1)$$

are

$$\zeta, \zeta^2, \ldots, \zeta^{\lfloor\frac{2p-1}{3}\rfloor}, \zeta, \ldots, \zeta^{\lfloor\frac{p-1}{3}\rfloor}$$

where $\zeta = \exp\left(\frac{2\pi i}{p}\right)$.

Proof

(a) $(V_{t,\mathbb{Q}})^{\sigma=1} = H^1(X_t,\mathbb{Q})^{\sigma=1} = H^1(X_t/\langle\sigma\rangle,\mathbb{Q})$

hence this follows as $X_t/\langle\sigma\rangle \cong \mathbb{P}^1$.

(b) ~~For example say $p=7$~~. This follows by exhibiting a $\mathbb{C}$-basis for $H^0(X_t,\Omega^1)$:

$$\frac{dx}{y^{p-1}}, \frac{dx}{y^{p-2}}, \dots \frac{dx}{y^{p-\lfloor\frac{2p-1}{3}\rfloor}} \quad \text{and}$$

$$\frac{x\,dx}{y^{p-1}}, \dots, \frac{x\,dx}{y^{p-\lfloor\frac{p-1}{3}\rfloor}}.$$ Then eg

$$\sigma^*\left(\frac{dx}{y^{p-1}}\right) = \frac{dx}{(\zeta y)^{p-1}} = \zeta\cdot\frac{dx}{y^{p-1}}, \text{ etc}$$

~~By the above each $V_{t,\mathbb{Q}}$ is a module over $\mathbb{Q}[\sigma]/(\sigma^{p-1}+\dots+1)\cong\mathbb{Q}(\zeta_p)$. hence a 2-dim vector space.~~

~~Proposition 2.1.2 For each $t$ there are $e_1, e_2 \in V_{\mathbb{Z}}$ such that $\{\sigma^a(e_b)\}_{a=1,\dots,p-2}$ span $V_{t,\mathbb{Q}}$ and such that~~

$$\cancel{\psi(\sigma^a(e_b), \sigma^{a'}(e_{b'})) = \sigma\ b}$$

## 2.2 PVHS's as above

Suppose we have a weight 1 rank $2(p-1)$ PVHS $(V_{\mathbb{Z}}, V^{1,0}_{S}, \psi)$ over $S = D$ and an automorphism $\sigma$ as above.

Lemma 2.2.1 There exist sections $e_1, e_2 \in V_{\mathbb{Z}}$ such that $\{\sigma^a(e_b)\}$ generate $V_{\mathbb{Q}}$.

(Pf) The system is constant and each $V_{s,\mathbb{Q}}$ is a $\mathbb{Q}[\sigma]/(\sigma^{p-1} + \cdots + 1) \cong \mathbb{Q}(\zeta)$ vector space, of dimension two. $\square$

Set for $j = 1, \ldots, p-1$

$$e_i^{(j)} = \sum_{a=0}^{p-1} \zeta^{-ja} \sigma^a(e_i) \quad \in V_{\mathbb{C}}$$

Then $\sigma(e_i^{(j)}) = \zeta e_i^{(j)}$ and $\overline{e_i^{(j)}} = e_i^{(p-j)}$.

Then

$$V_{\mathbb{C}} = \bigoplus_{j=1}^{p-1} V^{(j)}, \quad V^{(j)} = \mathbb{C} e_1^{(j)} \oplus \mathbb{C} e_2^{(j)}$$

and $V^{(j)}$ is the 2-dim$^l$ ~~g~~ sub system of $V_{\mathbb{C}}$ where $\sigma$ acts with eigenvalue $\zeta^j$. Since $\psi(\sigma x, \sigma y) = \psi(x, y)$ we have

- $\psi(V^{(j)}, V^{(j')}) = 0$, if $j + j' \neq p$

- $\psi$ perfect duality between $V^{(j)}$ and $V^{(p-j)}$

Conclusion 2.2.2

$$V_s^{1,0} = \bigoplus_{j=1}^{\lfloor\frac{p-1}{3}\rfloor} V_s^{(j)} \oplus \bigoplus_{j=\lfloor\frac{p-1}{3}\rfloor+1}^{\lfloor\frac{2p-1}{3}\rfloor} \mathbb{C}\cdot(e_1^{(j)}+\tau^{(j)}e_2^{(j)})$$

for certain $\tau^{(j)} = \tau^{(j)}(s) \in \mathbb{C}\cup\{\infty\} = \mathbb{P}^1(V^{(j)})$

The conditions for a PVHS:

(1.1.4) : $\tau^{(j)}(s)$ is holomorphic for $s \in D$

(1.1.5) : condition is void

(1.1.6.1) : $\psi\left(e_1^{(j)}+\tau^{(j)}e_2^{(j)},\, e_1^{(p-j)}+\tau^{(p-j)}e_2^{(j)}\right) = 0$

this means that $\tau^{(j)}$ determine $\tau^{(p-j)}$ for $j = \lfloor\frac{p-1}{3}\rfloor+1, \ldots \lfloor\frac{2p-1}{3}\rfloor$.

(1.1.6.2) : each $\tau^{(j)}(s)$ lies a (nonempty) open disc $\mathfrak{h}^{(j)} \subset \mathbb{P}^1(V^{(j)})$.

(Note: $\tau^{(j)} \in \mathfrak{h}^{(j)} \Leftrightarrow \tau^{(p-j)} \in \mathfrak{h}^{(p-j)}$.)

Conclusion 2.2.3

For $p=5$ (resp. $p=7$) the function $\tau^{(2)}: D \longrightarrow \mathfrak{h}^{(2)}$ (resp. $\tau^{(3)}$) determines the PVHS and so there is a "universal" PVHS's over $\mathfrak{h}^{(2)}$ (resp. $\mathfrak{h}^{(3)}$).

For $p \geq 11$ the "universal" variation lies over a product ~~$\mathfrak{h}$~~ of discs. $\geq 2$ oph.
(E.g. for $p=11$ $\mathfrak{h}^{(4)} \times \mathfrak{h}^{(5)}$.)

2.3 CM Hodge structures of Weight 1

Def. (2.3.1)
(a) An abelian variety $A/\mathbb{C}$ is said to have CM (complex multiplications) iff there exists a commutative subalgebra

$$K \subset \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with $[K:\mathbb{Q}] = 2\dim(A)$.

(b) For a wt 1 $\mathbb{Q}$-HS $V_{\mathbb{Q}}$ we ~~want~~ say it has CM if there exists

$$K \subset \mathrm{End}_{\mathbb{Q}\text{-Hodge}}(V_{\mathbb{Q}})$$

K commutative $[K:\mathbb{Q}] = \dim_{\mathbb{Q}}(V_{\mathbb{Q}})$.

<u>Proposition 2.3.2</u>. In the situation of 2.2.3, there is a dense set of $\tau^{(2)} \in \mathfrak{h}^{(2)}$ (with $p=5$ (resp $p=7$)) such that the associated Hodge structure has CM. [Note: for $p=11$ eg get <u>dense</u> in $\mathfrak{h}^{(4)} \times \mathfrak{h}^{(5)}$.]

<u>Proof</u> (Idea only). Take a self adjoint or anti-self adjoint $T \in \mathrm{End}_{\mathbb{Q}}(V_{\mathbb{Q}})$ commuting with $\sigma$ and let $\tau_T \in \mathbb{P}^1(V^{(2)})$ ~~be such that~~ be an eigenvecter for $T$. Then $K = \mathbb{Q}[\sigma, T] \subset \mathrm{End}(V_{\mathbb{Q}})$ will work.

Check (tricky) that $\{\tau_T\} \subset \mathbb{P}^1(V^{(2)})$ is dense.

$\square$

## 2.4 CM Jacobians

$\boxed{p=5}$ For any disc $D \subset S = \mathbb{P}^1 \setminus \{0,1,\infty\}$
we obtain $\tau^{(2)}: D \longrightarrow \mathfrak{h}^{(2)}$ as in 2.2.2

Claim 2.4.1
$\tau^{(2)}$ not constant! ~~(arguments same as in 1.2.5~~
(Arguments 1.2.5.1 or 1.2.5.3 works.)

So we conclude

Prop 2.4.2 There are infinitely many curves of genus 4 (resp 6) whose Jacobian has CM.

Conjecture 2.4.3 (André-Oort type Conj.)
There are only finitely many CM fibres in the families $X_t$ for $p \geq 11$.

Reference: de Jong & Noot, Jacobians with complex multiplication. (Texel)