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Project description

Compute the $\zeta^{(p)}(s,t)$ mentioned above, or make an educated guess, and verify that they satisfy the identities known for the $\zeta(s,t)$.

The "known identities" I have in mind are:

(a) Let Δ be the coproduct on $\mathbb{Q} \ll e_0, e_1 \gg$ for which the e_i are primitive: $\Delta e_i = e_i \otimes 1 + 1 \otimes e_i$. It induces a coproduct on $\mathbb{Q} \ll e_0, e_q \gg /\mathbb{Q} \ll e_0, e_1 \gg e_0 + e_1\mathbb{Q} \ll e_0, e_1 \gg$. In this quotient,

$$g_{\mathbb{R}} := \sum \zeta(s_1, \dots, s_r) e_0^{s_1 - 1} e_1 \dots e_0^{s_r - 1} e_1$$

is group like: $\Delta g_{\mathbb{R}} = g_{\mathbb{R}} \otimes g_{\mathbb{R}}$.

(b) $\zeta(s)\zeta(t) = \zeta(s,t) + \zeta(t,s) + \zeta(s+t) \quad$ (an instance of $\sum\limits_{n,m} = \sum\limits_{n>m} + \sum\limits_{m>n} + \sum\limits_{n=m}$)

The action of Frobenius is defined in [1], §11. A more natural definition is given in [2], but I don't expect it to make computations easier. The tangential base point required (tangent vector 1 at 0) is explained in [1], §15. A rather unsatisfactory computation of the $\zeta^{(p)}(s)$ is essentially the content of [1], 19.6, 19.7, and amounts to $\zeta^{(p)}(s)$ being the following regularization of

$$-p^s \sum_{p \nmid n} \frac{1}{n^s} \;\; :$$

write $\sum'$ for a sum extended only to n prime to p and for ℓ prime to p write formally

$$\ell^{1-s} {\sum}' \frac{1}{n^s} = \ell {\sum}' \frac{1}{(\ell n)^s} = \sum_{\alpha^\ell = 1} {\sum}' \frac{\alpha^n}{n^s} \;, \quad \text{hence}$$

$${\sum}' \frac{1}{n^s} = (\ell^{1-s} - 1)^{-1} \sum_{\alpha^\ell = 1, \alpha \neq 1} \; {\sum_{p \nmid n}}' \frac{\alpha^n}{n^s}.$$

It remains to regularize $\sum' \frac{z^n}{n^s}$ for z not congruent to $1 \mod p$. Let $(n \mod p^N)$ denote the residue of $n \mod p^N$. Define

$$\sum' \frac{z^n}{n^s} = \lim_N \sum' \frac{z^n}{(n \mod p^N)^s} = \lim_N \sum_k z^{p^N \cdot k} \sum_1^{p^N}{}' \frac{z^n}{n^s}$$
$$:= \lim_N (1 - z^{p^N})^{-1} \sum_1^{p^N}{}' \frac{z^n}{n^s}.$$

[**1**] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, in: Galois Groups over $\mathbb{Q}$, MSRI Publ. **16** (1989), p. 79–297.

[**2**] B. Chiarellotto and B. Le Stum, *F-isocristaux unipotents*, Comp. Math. **116** (1999), p. 81–110.