MODULAR FORMS OVER CM FIELDS

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Contents

1. Preparation

I will assume that the students will have some familiarity with the following (by now ancient) topics, though not necessarily with proofs:

(1) *Class field theory*, especially the formalism using the idele class groups, as summarized for example in [RaV], chapter 6;
(2) *Hecke characters* of type A_0 ([We]), and the analytic properties of their L-functions -as in section 7.4 of [RaV];
(3) *Classical modular forms* on the upper half plane, Hecke operators and the associated Galois representations; a good, relaxed survey is in [Ri]; and
(4) *Automorphic forms on $GL(2)$* from the representation theoretic point of view and the connection with (3), as in chapters 2-4 of [Ge], or selected portions of chapters 2-3 of [Bu]; [De] is another beautiful reference, but I am distressed by seeing the discrete group act there on the right (I guess I'm a closet leftist).

I would also assume that people have seen the rudiments of group cohomology, elliptic curves, cusps, $\cdots$. If all of this is mother's milk to someone, he/she could thumb through [Ti] for entertainment.

2. An outline of the course

The following should be taken as a sort of a rough set of goals for the course. In particular I am not sure if it will be possible to cover all the material, or even most of it. My main interest is to introduce the students to an active and interesting area without drowning them. (One of my favourite films is *Drowning by numbers* by Peter Greenaway.)

The main objects here will be analogs, for CM fields K of degree $2n$, of the classical modular forms of weight $k \geq 2$. (Recall that a CM field is a totally imaginary quadratic extension K of a totally real number field F.) These forms over K are not holomorphic, but they contribute to the cohomology of $Y_0(\mathfrak{A}) := \Gamma_0(\mathfrak{A})\backslash\mathcal{H}_3^n$ in degrees n and $2n$, where $\mathcal{H}_3$ is the hyperbolic 3-space and $\Gamma_0(\mathfrak{A})$ is a congruence subgroup of $\mathrm{SL}(2, \mathfrak{O}_K)$ associated to an an ideal $\mathfrak{A}$. There are also Hecke operators here and since they preserve the $\mathbb{Q}$-structure of the cohomology, one knows the algebraicity of the eigenvalues a_P of any cuspidal Hecke eigenform f. The cusp forms f in fact lie in a subspace called the *cuspidal cohomology* and a supplement is spanned by Eisenstein series. We will discuss briefly the work of Harder ([Ha]).

It is a difficult problem to know how many cusp forms there are of a given weight $k \geq 2$ over K. We will try to understand this. Over totally real number fields one can use the Riemann-Roch theorem (geometry) or the Selberg trace formula (analysis), but neither method applies here, the former because the space $Y_0(\mathfrak{A})$ has no complex structure and the latter because the archimedean component π_∞ of the automorphic representation π associated to a Hecke eigenform f is not isolated. (The non-isolation of the infinity type is also the reason why it is hard to construct Maass forms on the upper half plane; perhaps Sarnak will address this issue in his lectures.) The simplest way to get examples is to use algebraic Hecke characters of quadratic extensions M/K with the right infinity type. One can also base change *Hilbert modular forms* from the totally real subfield F to K, and then one can twist them by finite order characters to get new ones. So the problem becomes one of finding forms f not defined by Grossencharacters of quadratic extensions such that no abelian twist of f is a base change from F. Poincare series and *modular symbols* provide a few examples; see [EGM], [Cr] for the case when $[K : \mathbb{Q}] = 2$.

The *modularity conjecture* over K will say, in its simplest and perhaps the most important form, that any elliptic curve E over K is associated to a modular form f of weight 2, trivial character and $\mathbb{Q}$-coefficients. This can be verified for E of CM type, with f being a cusp form only when the complex multiplications are not defined over K. For $K = \mathbb{Q}[\sqrt{-3}]$, there is an explicit non-CM example in [Ta1], namely the curve E given by $y^2 + xy = x^3 + (3 + \sqrt{-3})x^2/2 + (1 + \sqrt{-3})x/2$, which does not come by base change from $\mathbb{Q}$. Of late I have been collaborating with Farshid Hajir, looking for such examples over quartic CM fields, and if we make sufficient progress before March 10, 2001, I will discuss such an example in the course. One important reason to find such an example is to deduce, when a successful matching is achieved, the Ramanujan bound for the coefficients of the associated form f, about which precious little is known outside the totally real situation. In any case, the existence of non-CM elliptic curves over K implies, at least conjecturally, the existence of cusp forms of weight 2 over K.

In the converse, so called *easy*, direction, it is conjectured that to every Hecke eigenform f over K with character ω one should be able to associate a 2-dimensional $\overline{\mathbb{Q}}_\ell$-representation $\rho_\ell = \rho_\ell(f)$ of $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ such that (for almost all primes P) the Hecke eigenvalue a_P of f equals the trace of the Frobenius Fr_P under ρ_ℓ, with $\det(\rho_\ell) = \omega\chi_\ell^{k-1}$, where χ_ℓ is the cyclotomic character. (Such a result is well known over totally real fields, but unlike in that case, one cannot expect that forms of weight 2, trivial character and $\mathbb{Q}$-coefficients give rise to elliptic curves E/K; this is because automorphic forms are not sensitive to coefficients.) The only published result so far in the non-totally real case is the theorem of R.L. Taylor ([Ta1]) for $[K : \mathbb{Q}] = 2$, giving the existence, for k even, of ρ_ℓ under mild (removable) hypotheses, but it yields the equality $a_P = \mathrm{tr}(\rho_\ell(Fr_P))$ only for P in a set of primes of density 1. This work depends on his earlier joint work with M. Harris and D.

Soudry ([HST]) giving a holomorphic lifting from $\mathrm{GL}(2)/K$ to $\mathrm{GSp}(4)/\mathbb{Q}$ whenever an appropriate central L-value is non-zero.

In the last couple of lectures, time permitting, I will outline an ongoing program to associate certain Galois representations to f over an arbitrary CM field K. First I will explain the *principle of functoriality*, which tells one when to expect liftings, and what sorts, between reductive groups, but I will do this by stressing mainly the concrete cases of GL(2), GL(4), GSp(4) and U(4). I will then indicate why it is difficult to blindly extend Taylor's work to the general CM K. In fact the method I will propose will utilize a tensor product construction ([Ra1,2]) and base change ([Cℓ1]), followed by a lifting to a holomorphic form g, say, on a suitable unitary group in 4 variables over F. Such a g will be *semi-regular*, the analog of a classical weight 1 form on the upper half plane. By multiplying by the ℓ^n-th power of a Hasse invariant form h, which is pulled back from large symplectic group (cf. section 3 of [BRa]), one constructs a family ϕ_n of forms of *regular type*, i.e., contributing to the cohomology of the associated Shimura variety, such that $g \equiv \phi_n \mod(\ell^{n+1})$. (There is a mistake in [BRa], but it does not affect this section.) Admitting some ramification conditions on f, one can then appeal to the results of Clozel ([Cℓ2]) and Kottwitz ([Ko]), and construct certain 4-dimensional representations $\tau_{\ell,n}$ of G_F. (Needless to say, the people who understand Clozel's course well will be happy when we get to this last part of our course. The earlier parts will be a lot easier to follow, however.) After that one needs to appeal to the theory of pseudo-representations of Wiles ([W]), as generalized by Taylor ([Ta2]), to piece together a representation associated to f - an instance of *vertical ℓ-congruences* in the terminology of [Ti].

Why does one have to go through so much trouble? The answer is simple. We need to use functoriality and move to a different group which has a Shimura variety so that one can use arithmetical algebraic geometry and/or congruences to produce ℓ-adic representations. The varieties attached to $\mathrm{GL}(2)/F$ or $U(4)/F$ for F totally real, and to $\mathrm{GSp}(4)/\mathbb{Q}$, are all nice because they are of *peℓ* type, i.e., they parametrize abelian varieties with *p*olarization, *e*ndomorphisms and *ℓ*evel structure, but the ones associated to $\mathrm{GSp}(4)/F$ are not so nice for $F \neq \mathbb{Q}$; the zeta functions, in fact even the congruence relations, are not understood for them. Admittedly this area is somewhat forbidding to get into, not because it is any harder than any other subject, but because it requires one to be comfortable with a variety of techniques. On the plus side, once you are in you will note that there is a host of unsolved problems.

3. **The student project**

As mentioned above, there is a construction in [Ta1] of a 2-dimensional Galois representation ρ_ℓ associated to a form π of weight $2k$ over an imaginary quadratic field K under a hypothesis on a non-vanishing L-value result, which has since been established ([FH]). But this article only proves that the Hecke eigenvalue a_P of π equals the trace of $\rho_\ell(Fr_P)$ at every prime P in a set of density 1. The reason for this is that he uses *congruence relations* for the variety Sh associated to $\mathrm{GSp}(4)/\mathbb{Q}$. ($Sh$ parametrizes polarized abelian surfaces with level structure.) Now there is a preprint [Weiss] of Weissauer giving the construction of 4-dimensional representations σ_ℓ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated to Siegel modular forms φ of weight ≥ 3. To be precise, there is a degree 4 polynomial $P_p(T)$ at an unramified prime

p associated to a Hecke eigenform φ (see section 2 of [BRa] for example), defining its L-function, and Weissauer's assertion is that the characteristic polynomial of $\sigma_\ell(Fr_p)$ equals $P_p(T)$. (You can just ADMIT this for the project.) There is another proof in the work of Laumon ([Lau]), though only for the case of constant coefficients (weight 3). The proof should be simpler for higher weights, but unfortunately this has not been done. The project will require the knowledge of Galois representations attached to Siegel modular forms of all weights ≥ 3, even if one begins with an f of weight 2 over K.

The project I propose is to substitute this result of Weissauer for (the weaker information on Galois representations coming from) congruence relations which Taylor uses, and obtain a ρ_ℓ which is correct at all the primes P which are unramified and prime to ℓ. This can be done within a week if one becomes familiar with the results of [Ta1] and an outline of the method. Analytically minded students could explain [FH] and indicate how and why it enters the picture. One could treat the results of [Weiss] (Theorem I on page 1) and [HST] as well as many of the arguments of [Ta1] as a big black box and try to get the desired result. Of course the idea is that one will learn some interesting stuff along the way. It will also be very useful to have this result available with complete reasoning for the general public.

I would like to meet with the few students doing the project with me on the first day of the winter meeting for half an hour or so, during which time I will give a sketch of how I think the verification and the writing process should go.

4. **Selected References**

These are given for general culture and for completeness. They are wide ranging and cover much more ground than we would need, and the students are NOT expected to know them for the lectures. The only exception is the stuff mentioned in the *Preparation* section and the references suggested in the project for those students doing it with me.

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