Notes on Kato-Siegel functions

A special function.

First consider the elliptic curve E_τ over $\mathbb{C}$ corresponding to the lattice $\mathbb{Z}+\tau\mathbb{Z}$, where $\mathrm{Im}(\tau) > 0$.

Consider the function

$$\Theta(u,\tau) = q^{\frac{1}{12}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \prod_{n>0}(1-q^n t)(1-q^n t^{-1}),$$

where $q = e^{2\pi i\tau}$ and $t = e^{2\pi i u}$. For any integer D prime to 6, construct the function

$$\theta_D^{E_\tau/\mathbb{C}} := (-1)^{\frac{D-1}{2}} \Theta(u,\tau)^{D^2} \Theta(Du,\tau)^{-1}.$$

Then $\theta_D^{E_\tau/\mathbb{C}}$ enjoys the following properties.

(i) $\theta_D^{E_\tau/\mathbb{C}}$ has divisor $D^2(e) - \ker[\times D]$.
(ii) For any isogeny $\alpha : E \to E'$ of degree prime to D between two such curves, $\alpha_* \theta_D^{E/\mathbb{C}} = \theta_D^{E'/\mathbb{C}}$.
(iv)
- $\theta_{-D} = \theta_D$ (This is obvious.)
- $\theta_1 = 1$
- $[\times M]_* \theta_{MC} = \theta_C^{M^2} \in \mathcal{O}^*(E - \ker[\times C])$ (In particular, $[\times D]_* \theta_D = 1$.)
- $\theta_C \circ [\times M] = \theta_{MC}/\theta_M^{C^2} \in \mathcal{O}^*(E - \ker[\times MC])$

(The reason for the numbering will be become apparent in a moment.)

Aim.

We wish to exhibit the algebraic nature of this phenomenon, and show that it generalizes to elliptic curves over arbitrary schemes, behaving well in families. Let $f : E \to S$ be an elliptic curve over an arbitrary scheme S. Let $\boldsymbol{\omega}_{E/S}$ be the invertible sheaf

$$f_* \Omega^1_{E/S} = e^* \Omega^1_{E/S},$$

which, since $\Omega^1_{E/S}$ is free along the fibres of f, is

$$= x^* \Omega^1_{E/S}$$

for any section $x \in E(S)$.

We want to assign to each E/S, and D prime to 6, a section

$$\theta_D^{E/S}$$

of $\mathcal{O}^*(E - \ker[\times D])$ such that

(i) $\theta_D^{E/S}$ (as a rational function) has divisor $D^2(e) - \ker[\times D]$.
(ii) The assignment is compatible with isogeny. Precisely, for any isogeny $\alpha : E \to E'$ of degree prime to D between elliptic curves over S, we have $\alpha_* \theta_D^{E/S} = \theta_D^{E'/S}$.

(iii) The assignment is compatible with base change. Precisely, for any base change

$$\begin{array}{ccc} E & \xleftarrow{\;g\;} & E \times_S S' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ S & \longleftarrow & S' \end{array}$$

we have $g^*(\theta_D^{E/S}) = \theta_D^{E\times_S S'/S'}$.

(iv)
- $\theta_{-D} = \theta_D$
- $\theta_1 = 1$
- $[\times M]_* \theta_{MC} = \theta_C^{M^2} \in \mathcal{O}^*(E - \ker[\times C])$ (In particular, $[\times D]_*\theta_D = 1$.)
- $\theta_C \circ [\times M] = \theta_{MC}/\theta_M^{C^2} \in \mathcal{O}^*(E - \ker[\times MC])$

(v) In the case where E is an elliptic curve over $\mathbb{C}$ corresponding to the lattice $\mathbb{Z} + \tau\mathbb{Z}$, $\mathrm{Im}(\tau) > 0$, the section $\theta_D^{E/\mathbb{C}}$ is as before. That is,

$$\theta_D^{E/\mathbb{C}} = (-1)^{\frac{D-1}{2}} \Theta(u,\tau)^{D^2} \Theta(Du,\tau)^{-1}.$$

Note that (i) and (ii) determine $\theta_D^{E/S}$ uniquely. (By (i) any other possible assignment is $u\theta_D^{E/S}$, for some $u \in \mathcal{O}^*(S)$. Applying (ii) with $\alpha = [\times 2]$ and $[\times 3]$ (2 and 3 are prime to D) gives

$$\begin{aligned} & u\theta_D^{E/S} \\ =& \alpha_*(u\theta_D^{E/S}) \\ =& \alpha_*(u)\alpha_*\theta_D^{E/S} \\ =& \alpha_*(u)\theta_D^{E/S} \\ =& \text{ both } u^4\theta_D^{E/S} \text{ and } u^9\theta_D^{E/S}, \end{aligned}$$

whence $u^4 = u = u^9, \Rightarrow u = 1$.

Example. Suppose $S = \operatorname{Spec} k$, k algebraically closed, $\alpha : E \to E'$ separable. Then (ii) becomes

$$\prod_{\substack{x\in E(k)\\ \alpha(x)=y}} \theta_D^{E/k}(x) = \theta_D^{E'/k}(y) \qquad \forall y \in E'(k),$$

which is the familiar distribution relation.

Proof of main result. First note that if $S = \operatorname{Spec} k$, then $\ker[\times D] - D^2(e)$ is principal. (Because D is odd, the sum of $\ker[\times D]$ on the elliptic curve is 0.)

- To give a rule θ_D satisfying (i) and (iii) is equivalent to giving an isomorphism of line bundles

$$\mathcal{O}_E(\ker[\times D]) \to \mathcal{O}_E(D^2 e)$$

compatible with base change. Taking any fibre of E/S, we have the situation $S = \operatorname{Spec} k$ above; hence the line bundles are indeed isomorphic when restricted to any fibre. Then, our task is equivalent to finding, for each E/S, a trivialization of

$$e^*\mathcal{O}_E(\ker[\times D]) \otimes e^*\mathcal{O}_E(D^2 e)^\vee,$$

which is

$$\begin{aligned}
&= e^*\mathcal{O}_E(\ker[\times D]) \otimes e^*\mathcal{O}_E(-D^2 e) \\
&= e^*[\times D]^*\mathcal{O}_E(e) \otimes e^*\mathcal{O}_E(-D^2 e) \\
&= e^*\mathcal{O}_E(e)^{\otimes(1-D^2)} \\
&= \boldsymbol{\omega}_{E/S}^{\otimes(D^2-1)},
\end{aligned}$$

compatible with base change.

Now we know the set of nowhere-vanishing sections of $\boldsymbol{\omega}^{\otimes 12d}$ over the moduli stack – that is, the collections, of a section of $\boldsymbol{\omega}_{E/S}^{\otimes 12d}$ for each E/S, compatible with base change and isogeny – is $\{\pm\Delta^d\}$, for any $d \in \mathbb{Z}$, where Δ is the discriminant. Then (since $(D,6) = 1 \Rightarrow D \equiv 1 \mod 12$) we have 2 non-vanishing sections of $\boldsymbol{\omega}^{\otimes(D^2-1)}$, that is, $\pm\Delta(E/S)^{(D^2-1)/12}$. Let $\pm\phi_D^{E/S}$ be the corresponding functions on $E - \ker[\times D]$. So both $E/S \mapsto \pm\phi_D^{E/S}$ satisfy (i) and (ii).

Change base so that α factors as a product of isogenies of prime degree. Thus to verify (ii) we may assume $\deg\alpha = p$ prime.

The quotient $g_p(E/S,\alpha) := \alpha_*\phi^{E/S}(\phi^{E'/S})^{-1} \in \mathcal{O}^*(S)$ is compatible with base change. The modular stack $\mathcal{M}_{\Gamma_0(N)}$ classifies pairs $(E/S,\alpha)$ where $\alpha : E \to E'$ is a cyclic isogeny of degree N. So g_p is a modular unit $\in \Gamma(\mathcal{M}_{\Gamma_0(p)}, \mathcal{O}^*)$, and so $g_p(E/S,\alpha) = \pm 1 \;\; \forall (E/S,\alpha)$. And the sign depends only on p.

- To determine the sign evaluate $g_p(E/\mathbb{F}_p, \mathrm{Fr}_E)$. Now $\mathrm{Fr}_{E*} : \kappa(E)^* \to \kappa(E)^*$ is the norm map. So for p odd, $g_p(E, \mathrm{Fr}_E) = 1$. In particular this does not depend on our choice of $\pm\phi_p^{E/S}$. For $p = 2$, though, replacing one by the other replaces g_2 by $-g_2$. Therefore exactly one of $\pm\phi^{E/S}$ will make θ_2 satisfy (ii).

- We check (iv). θ_{-D} also satisfies (i) and (ii), which uniquely determine θ_D. So $\theta_{-D} = \theta_D$. The constant 1 satisfies (i) and (ii) for $D = 1$; hence $\theta_1 = 1$.

$[\times M]_*\theta_{MC}$ has divisor $M^2(C^2(e) - \ker[\times C])$ and is compatible with base change, so $[\times M]_*\theta_{MC} = \epsilon\theta_C^{M^2}$, $\epsilon = \pm 1$. Now (ii) gives

$$\begin{aligned}
\epsilon\theta_C^{M^2} &= [\times M]_*\theta_{MC} \\
&= [\times M]_*[\times 2]_*\theta_{MC} \\
&= [\times 2]_*[\times M]_*\theta_{MC} \\
&= [\times 2]_*(\epsilon\theta_C^{M^2}) \\
&= \epsilon^4\theta_C^{M^2} \\
&= \theta_C^{M^2}
\end{aligned}$$

and so $\epsilon = 1$.

Using $D = M$ and $C = 1$ produces $[\times D]_* \theta_D = \theta_1^{D^2} = 1$.

Now $\theta_C \circ [\times M]$ and $\theta_{MC}/\theta_M^{C^2}$ both have divisor

$$C^2 \ker[\times M] - \ker[\times MC];$$

hence their ratio is a unit compatible with base change. So (i) and (ii) give the result as before.

- A final matter is to check that condition (v) holds – that is, that our $\theta_D^{E/S}$ is indeed a generalization of the analytic $\theta_D^{E_\tau/\mathbb{C}}$ given at first. This found by calculating that $F(u,\tau) := \Theta(u,\tau)^{D^2}\Theta(Du,\tau)^{-1}$ is a function on E_τ ($\approx \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$) with divisor $D^2(e) - \ker[\times D]$, and which is $\mathrm{SL}_2(\mathbb{Z})$-invariant. Then $F(u,\tau)$ is a constant multiple of $\theta_D^{E/S}$ for $E = E_\tau$, the constant being independent of τ.

Consideringt a curve E_τ defined over $\mathbb{R}$ with two real connected components (for example $Y^2 = X^3 - X$, where $\tau = i$) we calculate that

$$[\times 2]_* F(u,\tau) = (-1)^{\frac{D-1}{2}} F(u,\tau).$$

□