p-adic Modular Forms

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3/13/01

Let $E/R/R_0$ be an elliptic curve over an R_0-algebra R, where $R_0 = \mathcal{O}_K$ with $[K : \mathbb{Q}_p] < \infty$. Now consider E/K, then we have two cases:

$$v(E) \in \begin{cases} \text{not defined} & \text{if } E \text{ is very supersingular} \\ [0,1) \cap \mathbb{Q} & \text{otherwise} \end{cases} \tag{1}$$

Theorem 1. *(Katz-Lubin) If*

$$v(E) < \begin{cases} \frac{p}{p+1} & \textit{if } p \geq 5 \\ \frac{p}{2(p+1)} & \textit{if } p = 3 \\ \frac{p}{4(p+1)} & \textit{if } p = 2 \end{cases} \tag{2}$$

then E has a "canonical" subgroup of ord=p.

Remark 1. $v(E) = 0 \Leftrightarrow E$ *has ordinary reduction, and then the canonical subgroup is just the kernel of the reduction map on its p-torsions.*

Assume $v(\rho) < c_p$, where c_p denotes the number on the right of (2) corresponding to different p's. If $(E/R, \omega, Y)$ is a ρ-overconvergent test object, then $v(E_K) \leq v(\rho) < c_p$. So E has a canonical subgroup H, and $(E/R, \omega, H)$ is a classical test object plus a subgroup of order p. A rule on these objects is a classical modular form of level p. Hence we get a map from classical modular forms of level p over K_0 to ρ-overconvergent forms of level 1. So we also have a U_p operator acting on the ρ-overconvergent forms. If f is a ρ-overconvergent, then

Remark 2. *Let E/K have $v(E) < c_p$, and H be the canonical subgroup, then*
(1) If C is a subgroup of order n with $(n, p) = 1$ then $v(E/C) = v(E)$,
(2) If C is not canonical then $v(E/C) = \frac{1}{p}v(E)$,
(3) If $v(E) < \frac{1}{p}c_p$ then $v(E/C) = pv(E)$, so in fact U_p maps ρ-overconvergent forms to ρ^P-overconvergent forms.

Definition 1.

$$\mathbb{M}_k(K_0, \rho) = (\rho - \text{overconvergent forms of weight } k \text{ defined over } R_0) \otimes K_0.$$

Then $\mathbb{M}_k(K_0, \rho)$ is a p-adic Banach space over K_0.

As the remark indicates, we will have Hecke operators T_l for $l \neq p$ acting on $\mathbb{M}_k(K_0, \rho)$, and U_p: $\mathbb{M}_k(K_0, \rho) \to \mathbb{M}_k(K_0, \rho^p)$.

While at the same time there is a natural inclusion

$$\mathbb{M}_k(K_0, \rho^p) \longrightarrow \mathbb{M}_k(K_0, \rho)$$

where $v(\rho) < \frac{1}{p}c_p$.

Hence we get a map

$$U_p : \mathbb{M}_k(K_0, \rho) \longrightarrow \mathbb{M}_k(K_0, \rho)$$

One can also get $U_p(\sum a_n q^n) = \sum a_{np} q^n$.

Remark 3. *T_l's are continuous. U_p is even better than that! Let V be a big infinite dimensional p-adic Banach space, and assume $e_1, e_2, \dots$ is a countable Banach basis of V. Then every $v \in V$ can be written uniquely as*

$$v = \sum a_i e_i, \text{ with } a_m \to 0,\ a_n \in K_0$$

Let $T : V \to V$ be a continuous operator, and $T(e_i) = \sum c_{ji} e_j$. So c_{ji} is the matrix of T with respect to the basis. Then the queation is: does this matrix have a trace? Of course one cannot expect an affirmative answer in general as the identity matrix has no trace.
But the operator $T : e_i \to p^i e_i$ of V has a trace$=\sum p^i = \frac{p}{1-p}$.

Now denote $\mathcal{L}(V, V)$=continuous linear maps:$V \to V$. $\mathcal{L}(V, V)$ inherits a norm from V. Let F be the subspace consisting of the maps whose image is finite dimensional. We define compact operators to be the closure of these F's.

Compact operators have traces, and even better, they have a spectral theory. Now say C is a compact linear operator, i.e. $C = \lim_{n\to\infty} C_n$, where $C_n : V \to V$ have finite dimensional images. Put

$$P_n(X) = \det(I - XC_n) = 1 - t_n X + \cdots + (-1)^n \det(C_n) X^n$$

then P_n's converge to a power series $P \in K_0[[X]]$ called the characteristic power series of C.

Example: Let $C_n = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $C = \lim C_n$. Then

$$P_n(X) = \prod_{i=1}^{n} (1 - p^i X)$$

therefore

$$P(X) = \prod_{i=1}^{\infty} (1 - p^i X) \in K_0[[X]]$$

and $P(x)$ converges ofor any $x \in K_0$.

Now we have a very nice result

Theorem 2. *If $v(\rho) \in (0, \frac{1}{p}c_p)$, then $U_p : \mathbb{M}_k(K_0, \rho) \longrightarrow \mathbb{M}_k(K_0, \rho)$ is compact.*

Re-interpretation of G-M: Fix ρ such that $0 < v(\rho) < \frac{1}{p}c_p$. Recall that $M_k(\Gamma_0(p), K_0)$ denotes the classical modular forms with weight k of level p over K_0. Then we have a U_p-covariant linear injection

$$M_k(\Gamma_0(p), K_0) \longrightarrow \mathbb{M}_k(K_0, \rho)$$

$M_k(\Gamma_0(p), K_0) =$ (old part) $\oplus$ (new part). U_p acts differently on these two parts:
(1) if $f \in$(old part), then $U_p(f) = a_p f$ and U_p has eigenvalues as roots of $X^2 - a_p X + p^{k-1}$, both of which have valuation$\leq k-1$,
(2) if $f \in$(new part), then U_p has eigenvalues $\pm p^{\frac{p-2}{2}}$. Therefore if λ is a U_p-eigenvalue on the classical forms, then $v(\lambda) \leq k-1$. The converse is almost true!

Theorem 3 (Coleman). *Assume $f \in \mathbb{M}_k(K_0, \rho)$ is an eigenform for U_p, T_l, and the U_p-eigenvalue is λ. If $v(\lambda) < k - 1$ then $f \in$ the image of $M_k(\Gamma_0(p), K_0)$.*

Definition. $v(\lambda)$ is called the slope of the overconvergent form f.

Hence one can retrieve classical forms as being "overconvergent forms of small slope".

Gouvea-Mazur Conjecture. Let $k \in 2\mathbb{Z}$, $\alpha \in \mathbb{Q}$, $\mathbb{M}_k(K_0, \rho)$, and $d(k, \alpha) = \sharp\{\text{eigenvalues of } U_p \text{ with valuation } \alpha\}$. Then $k_1 \equiv k_2 \pmod{(p-1)p^m}$, for $m \geq \alpha$, will imply that $d(k_1, \alpha) = d(k_2, \alpha)$.

Theorem 4 (Coleman). *If $P_k(X)$=char power series of U_p acting on $\mathbb{M}_k(K_0, \rho)$, then P_k varies analytically with k.*

This theorem implies that $d(k, \alpha)$ is a "locally constant" function of k.

Proposition 2. *If $k_1 \equiv k_2$ (mod $(p-1)p^m$), and $\alpha < O(\sqrt{m})$, then $d(k_1, \alpha) = d(k_2, \alpha)$.*

Example of the Spectrum of U_p.

Let's seek the structure of U_2 on $\mathbb{M}_0(K_0, \rho)$ (i.e. $k = 0, N = 1$). Let the char power series of U_2 be

$$\sum_{n \geq 0} a_n X^n = \prod_{i \geq 0} (1 - \lambda_i X).$$

The question is: what are the valuations of λ_i?

Inspired by a method of Kilford, we find that:

Theorem 5. *(Buzzard, Calegari) The valuations are 3,7,13,15,17,..., where the ith term is given by*

$$1 + 2v_2\left(\frac{(3i)!}{i!}\right).$$

Proof. Let's write down a basis for $\mathbb{M}_0(K_0, \rho)$ (the basis depends on ρ although the characteristic p.s. of ρ does not), say,

$$1, \alpha f, \alpha^2 f^2, \alpha^3 f^3, \cdots$$

where

$$f = \frac{\Delta(q^2)}{\Delta(q)} = q + 24q^2 + \cdots$$

and $\alpha = \alpha(\rho), \alpha \in \bar{\mathbb{Q}}_2, |\alpha| < 1$.

The matrix of U_2 is:

$$U_2(f^m) = \sum_{n = \lceil \frac{m}{2} \rceil}^{2m} s_{m,n} f^n$$

where

$$s_{m,n} = 2^{8n-4m-1} \cdot 3m(m+n-1)!/(2n-m)!(2m-n)!$$

Write $U_2 = A \cdot B$, where A is lower triangular, B is upper triangular, with 1's on both diagonals. Actually we can compute the entries A_{ij} and B_{ij}.

Now let $A = C \cdot D$ with D diagonal, then

$$D_{ii} = 2^{1+2v((3i)!/i!)}$$

Once we take $\alpha = 2^6$: it changes C_{ij} and B_{ij} by $2^{6(j-i)}$. Then the following lemma concludes the proof. □

Lemma 3. *After making the change if $C \equiv B \equiv Id \mod 2$, then the slopes of the characteristic power series of U_2 and D are the same.*